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Twisted Dual-Group Algebras: Equivariant Deformations of $C_0(G)$

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We consider a family of twisted Fourier algebras $A(G, \omega)$ of a locally compact group G, which in the case of a abelian group G are the Fourier transforms of the usual twisted group algebras of $\hat{G}$. The corresponding C^*-algebras $C^*(\hat{G}, \omega)$ are deformations of $C_0(G)$, which are equivariant in the sense that G still acts by left translation. The main examples come from cocycles σ on the dual of an abelian subgroup H of G; we prove that for such cocycles the twisted dual-group algebras $C^*(\hat{G}, \omega)$ are induced from the twisted group algebras $C^*(\hat{H}, \sigma)$, and we give detailed formulas for the multiplication on $A(G, \omega)$ which extend to larger dense subalgebras of $C_0(G)$ and $C_b(G)$. We anticipate that these larger subalgebras will be useful for constructing deformations of homogeneous spaces $C_0(G/\Gamma)$.

It is standard practice to deform the multiplication on a group algebra using a 2-cocycle σ on the group: the group algebra is spanned by a copy $\{\delta_x : x \in G\}$ of the group G, with the multiplication defined by $\delta_x \delta_y = \delta_{xy}$, and the *twisted group algebra* is spanned by a family $\{\delta_x\}$ satisfying $\delta_x \delta_y = \sigma(x, y)\, \delta_{xy}$. This works for discrete groups, purely algebraically, and for locally compact groups, where one uses either the L^1-group algebra $L^1(G)$ or the group C^*-algebra $C^*(G)$. Many of the recent constructions of non-commutative spaces have been obtained by recognising

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the space as the dual of a locally compact group $\hat{G}$, and then deforming $C_0(G) \cong C^*(\hat{G})$ into a twisted group algebra $C^*(\hat{G}, \sigma)$: thus, for example, the non-commutative tori are by definition the twisted group algebras $C^*(\mathbf{Z}^n, \sigma)$. Of course, this works only if the underlying group G is abelian. Here we shall describe a method of directly deforming $C_0(G)$ for arbitrary G, using the dual cocycles introduced by Wassermann and Landstad in their classification of ergodic actions of compact groups on von Neumann algebras [23, 12]. Our deformations are equivariant, in the sense that G acts by left translation on each of the deformed algebras; they are not quantum groups, although similar techniques can be used to deform $C_0(G)$ into quantum groups [13].

Just as deformations of $C^*(\hat{G})$ are defined first on $L^1(\hat{G})$, we begin with the Fourier algebra $A(G) \subset C_0(G)$. If we view $A(G)$ as the predual of the von Neumann algebra $\mathscr{L}(G)$, the usual pointwise multiplication on $A(G)$ can be described in terms of the comultiplication $\delta_G: \mathscr{L}(G) \to \mathscr{L}(G) \otimes \mathscr{L}(G)$ by $\langle T, fg \rangle = \langle \delta_G(T), f \otimes g \rangle$ for $T \in \mathscr{L}(G)$. A *dual cocycle* is a unitary element ω of the von Neumann algebra tensor product $\mathscr{L}(G) \otimes \mathscr{L}(G)$ such that

$$(\omega \otimes 1)\, \delta_G \otimes i(\omega) = (1 \otimes \omega)\, i \otimes \delta_G(\omega) \qquad \text{in} \quad \mathscr{L}(G) \otimes \mathscr{L}(G) \otimes \mathscr{L}(G),$$

and the *twisted Fourier algebra* $A(G, \omega)$ is then the usual Banach space $A(G)$ with multiplication defined by

$$\langle T, f *_\omega g \rangle = \langle \delta_G(T)\, \omega^*, f \otimes g \rangle \qquad \text{for} \quad T \in \mathscr{L}(G).$$

In Section 1, we explain carefully how, for an abelian group G, dual cocycles are the Fourier transforms of ordinary cocycles on $\hat{G}$. We then discuss the extra normalisation condition we need to ensure that $A(G, \omega)$ is a $*$-algebra with the usual involution: this is one of the technical problems which arises in extending the basic theory of [23, 12] from compact to locally compact groups. The algebra $A(G, \omega)$ has a regular representation on $L^2(G)$, and the *twisted dual-group algebra* $C_r^*(\hat{G}, \omega)$ is by definition the closure of $A(G, \omega)$ in $B(L^2(G))$. (For technical reasons, we have chosen to work with this algebra rather than the enveloping algebra $C^*(A(G, \omega))$, but we conjecture that they coincide.) The action of G by left translation on $A(G)$ extends to an action $\alpha: G \to \operatorname{Aut} C_r^*(\hat{G}, \omega)$, so that we have constructed equivariant deformations of $C_0(G) = C_r^*(\hat{G}, 1 \otimes 1)$.

The main examples of dual cocycles are the Fourier transforms of ordinary cocycles on abelian groups, and those *inflated* from these dual cocycles on abelian subgroups: if H is a subgroup of G, $\mathscr{L}(H)$ embeds in $\mathscr{L}(G)$, and a dual cocycle ω on H gives a dual cocycle $\operatorname{Inf}_H^G \omega$ on G (Proposition 3.2). (There is some evidence that all dual cocycles arise this way; see the comments at the beginning of Section 4, and the Appendix.) Our main theorem says that the twisted dual-group algebra $C_r^*(\hat{G}, \mathrm{Inf}\,\omega)$ is then determined by $C_r^*(\hat{H}, \omega)$ and the action α of H by left translation, as an induced C^*-algebra:

$$C_r^*(\hat{G}, \mathrm{Inf}_H^G\,\omega) \cong \mathrm{Ind}_H^G\,(C_r^*(\hat{H}, \omega), \alpha);$$

curiously, this appears to be an interesting result even for abelian groups (see Remark 3.4). The proof of this theorem in Section 3 requires some basic properties of induced C^*-algebras and their representations, which we present in Section 2. In particular, we show there that if (π, S) is a covariant representation of (B, β), then there is a natural representation $\tilde{\pi}$ of $\mathrm{Ind}(B, \beta)$ on the Hilbert space of the induced representation $\mathrm{Ind}\,S$.

Our twisted dual-group algebras, then, provide a family of deformations of $C_0(G)$. We would like to extend our construction to give deformations of homogeneous spaces $C_0(G/\Gamma)$, at least in examples like those considered by Rieffel [21]. In order to preserve the left action of G on such a deformation, we want Γ to act by right translation on the algebras $C_r^*(\hat{G}, \omega)$, and then the fixed-point algebras $C_r^*(\hat{G}, \omega)^\Gamma$ should be deformations of $C_0(G/\Gamma)$. Now there certainly are circumstances—including, encouragingly, those studied in [21]—in which such an action of Γ exists (see the end of Section 4), but in general the fixed-point algebra will be trivial. Rieffel has found one way around this which involves taking a generalised fixed-point subalgebra in the multiplier algebra [22]. We propose an alternative approach, in which we directly enlarge the subalgebra of $C_0(G)$ to include bounded functions on G which are invariant under Γ. To this end, we work out integral formulas for the multiplication on $A(G, \mathrm{Inf}\,\omega)$, in the case where ω is inflated from an abelian subgroup H and the original cocycle on $\hat{H}$ has an explicit, but common, form. We do this in Section 4, and then in Section 5 discuss a particular dense subalgebra $C_{0,1}(G)$ of $C_0(G)$, which can be readily extended to a much larger space of bounded continuous functions on which our formulas still make sense. We plan to describe deformations of $C_0(G/\Gamma)$ and their properties in a subsequent article.

Notation. We denote by L^G and R^G, respectively, the left and right regular representations of a locally compact group G on $L^2(G)$; we shall use left Haar measure throughout, so that R^G is defined by $R_x^G(\xi)(y) = \Delta(x)^{1/2}\,\xi(yx)$, where Δ is the modular function of G. We write $\mathcal{L}(G)$ for the von Neumann algebra $L^G(G)''$, and $\otimes$ for the von Neumann algebra tensor product. Then $\delta_G\colon \mathcal{L}(G) \to \mathcal{L}(G) \otimes \mathcal{L}(G)$ is the comultiplication on $\mathcal{L}(G)$: on the one hand, if W_G is the unitary operator on $L^2(G) \otimes L^2(G) = L^2(G \times G)$ defined by $W_G\xi(x, y) = \xi(x, x^{-1}y)$, then $\delta_G(T) = W_G(T \otimes 1)W_G^*$, and on the other hand, we can view δ_G as the unique normal homomorphism such that $\delta_G(L_x^G) = L_x^G \otimes L_x^G$. We write Σ for the flip automorphism of $\mathscr{L}(G) \otimes \mathscr{L}(G)$, 1 for the identity of an algebra, and i for the identity mapping between algebras. We shall freely use subscript notation for operators acting on tensor product spaces: thus, for example, if $T \in \mathscr{L}(G) \otimes \mathscr{L}(G)$, then T_{23} stands for $1 \otimes T$, or T_{13} for $\sum \otimes i(1 \otimes T) = i \otimes \sum(T \otimes 1)$. We use $\langle \cdot, \cdot \rangle$ for the pairing between a von Neumann algebra and its predual, and for the pairing between an abelian group and its dual. If f: $T \mapsto (T\xi \mid \eta)$ is a vector functional on a von Neumann algebra $M \subset B(\mathscr{H})$ (we sometimes write $f = \omega_{\xi, \eta}$), then S_f denotes the slice map S_f: $N \otimes M \to N$ characterised by

$$(S_f(T)\, h \mid k) = (T(h \otimes \xi) \mid k \otimes \eta) = (T \mid \omega_{\xi, \eta} \otimes f).$$

Thus, for example, since every functional $f \in A(G) = \mathscr{L}(G)_*$ is a vector functional, we obtain slice maps S_f: $N \otimes \mathscr{L}(G) \to N$.

1. Dual Cocycles and Twisted Dual-Group Algebras

By a cocycle on a locally group G, we shall mean a Borel 2-cocycle with values in $\mathbf{T}$; that is, a Borel function σ: $G \times G \to \mathbf{T}$ satisfying

$$\sigma(r+s)\, \sigma(r+s, t) = \sigma(s, t)\, \sigma(r, s+t) \qquad \text{for} \quad r, s, t \in G. \tag{1.1}$$

(We use additive notation because all our cocycles will be on abelian groups or their duals.) The *twisted group algebra* $L^1(G, \sigma)$ is the Banach space $L^1(G)$ equipped with the multiplication and involution

$$\begin{aligned} f *_\sigma g(t) &= \int f(s)\, g(t-s)\, \sigma(s, t-s)\, ds \\ f^*(t) &= \Delta(t)^{-1}\, \overline{\sigma(t, -t)}\, \overline{f(-t)}. \end{aligned} \tag{1.2}$$

The *twisted group C^*-algebra* $C^*(G, \sigma)$ is the enveloping C^*-algebra of $L^1(G, \sigma)$. (See [7, p. 417] for a discussion of enveloping algebras which does not assume bounded approximate identities, or [16] for an alternative approach.)

Definition 1.1. A *dual cocycle* on a locally compact group G is a unitary $\omega \in \mathscr{L}(G) \otimes \mathscr{L}(G)$ satisfying

$$\omega_{12}(\delta_G \otimes i(\omega)) = \omega_{23}(i \otimes \delta_G(\omega)) \qquad \text{in} \quad \mathscr{L}(G) \otimes \mathscr{L}(G) \otimes \mathscr{L}(G). \tag{1.3}$$

To justify this definition, note that if G is abelian, we can view a cocycle σ on $\hat{G}$ as a unitary element of $L^\infty(\hat{G}) \otimes L^\infty(\hat{G}) = L^\infty(\hat{G} \times \hat{G})$. Since the Fourier transform of $\mathscr{L}(G)$ onto $L^\infty(\hat{G})$ converts δ_G into the comultiplication $\alpha_G \colon L^\infty(\hat{G}) \to L^\infty(\hat{G} \times \hat{G})$ satisfying $\alpha_G(f)(s, t) = f(s+t)$, and the cocycle identity (1.1) for σ can be written

$$\sigma_{12} \alpha_G \otimes i(\sigma) = \sigma_{23} i \otimes \alpha_G(\sigma),$$

the Fourier transform $\omega \in \mathscr{L}(G) \otimes \mathscr{L}(G)$ of $\sigma \in L^\infty(\hat{G}) \otimes L^\infty(\hat{G})$ is a unitary operator satisfying (1.3).

The twisted dual-group algebra will be based on a twisted version of the Fourier algebra $A(G)$ [6]. Given a dual cocycle ω on G and $f, g \in A(G)$, we define $f *_\omega g \in A(G) = \mathscr{L}(G)_*$ by

$$\langle T, f *_\omega g \rangle = \langle \delta_G(T)\, \omega^*, f \otimes g \rangle \qquad \text{for} \quad T \in \mathscr{L}(G); \qquad (1.4)$$

since $T \to \delta_G(T)\, \omega^* = W_G(T \otimes 1)\, W_G^* \omega^*$ is weakly continuous, the right-hand side is weakly continuous in $T \in \mathscr{L}(G)$, and there is a unique functional $f *_\omega g \in \mathscr{L}(G)_*$ satisfying (1.4). To motivate (1.4), suppose G is abelian, ω is the Fourier transform of a cocycle σ on $\hat{G}$, and f, g are the (inverse) Fourier transforms of $h, k \in L^1(\hat{G})$, so, for example, $f(x) = \mathscr{F}h(x) = \int \langle x, s \rangle\, h(s)\, ds$. Then

$$f *_\omega g(x) = \langle L_x, f *_\omega g \rangle = \langle \delta_G(L_x)\, \omega^*, \hat{h} \otimes \hat{k} \rangle.$$

Under the Fourier transform L_x goes to the function $f_x \colon t \mapsto \overline{\langle x, t \rangle}$, and $\delta_G(L_x)\, \omega^*$ to the function $(s, t) \mapsto \overline{\langle x, s+t \rangle\, \sigma(s, t)}$. Thus

$$\begin{aligned} \langle L_x, f *_\omega g \rangle &= \iint \overline{\langle x, s+t \rangle\, \sigma(s, t)}\, h(s)\, k(t)\, ds\, dt \\ &= \int \overline{\langle x, t \rangle} \left(\int \overline{\sigma(s, t-s)}\, h(s)\, k(t-s)\, ds \right) dt \\ &= \langle f_x, h *_{\bar{\sigma}} k \rangle. \end{aligned}$$

Since we know that $\mathscr{F}$ is an isometric Banach space isomorphism of $L^1(\hat{G})$ onto $A(G)$, this proves that the Fourier transform induces an isometric Banach algebra isomorphism of $L^1(\hat{G}, \bar{\sigma})$ onto $(A(G), \omega)$.

LEMMA 1.2. *For an arbitrary locally compact group G, the Banach space $A(G)$ is a Banach algebra with respect to the multiplication $*_\omega$ defined by* (1.4); *we denote this Banach algebra by $A(G, \omega)$.*

Proof. The various distributive laws seem to be clear, so we have to check associativity and submultiplicativity of the norm. But the first follows from the coassociativity of δ_G and the cocycle identity (1.3),

$$\begin{aligned}(f *_\omega g) *_\omega h(x) &= \langle \delta_G \otimes i(\delta_G(L_x)\,\omega^*)\,\omega_{12}^*, f \otimes g \otimes h\rangle \\ &= \langle \delta_G \otimes i(\delta_G(L_x))(\omega_{12}\,\delta_G \otimes i(\omega))^*, f \otimes g \otimes h\rangle \\ &= \langle i \otimes \delta_G(\delta_G(L_x))(\omega_{23}\, i \otimes \delta_G(\omega))^*, f \otimes g \otimes h\rangle \\ &= \langle i \otimes \delta_G(\delta_G(L_x)\,\omega^*)\,\omega_{23}^*, f \otimes g \otimes h\rangle \\ &= f *_\omega (g *_\omega h)(x),\end{aligned}$$

and the second is easy. (The comultiplication δ_G is isometric, $\|\omega\| = 1$, and $\|f \otimes g\| = \|f\|\,\|g\|$.)

To define the involution on $A(G, \omega)$ analogous to that in (1.2), we would need a non-commutative (variable-free) analogue of the factor $\sigma(t, -t)$. The function $t \mapsto \sigma(t, -t)$ can be recovered from a σ-representation $V\colon \hat{G} \to \mathscr{H}$ by comparing V_t with V_{-t}:

$$V_t V_{-t} = \sigma(t, -t)\, V_{t-t} = \sigma(t, -t)\, 1.$$

If we view V as the element $t \mapsto V_t$ of $B(\mathscr{H}) \otimes L^\infty(\hat{G}) \simeq L^\infty(\hat{G}, B(\mathscr{H}))$, then $t \mapsto V_{-t}$ is $i \otimes \mu(V)$, where μ is the homomorphism on $L^\infty(\hat{G})$ given by $\mu(f)(t) = f(-t)$, and then $u\colon t \mapsto \sigma(t, -t)$ is characterised by $i \otimes \mu(V)\, V = 1 \otimes u$. The Fourier transform of μ is the linear map $\nu\colon \mathscr{L}(G) \to \mathscr{L}(G)$ such that $\nu(L_x) = L_{x^{-1}}$; this always makes sense—it can be defined spatially by $(\nu(T)\, h \mid k) = (T\bar{k} \mid \bar{h})$ for $h, k \in L^2(G)$—but is anti-multiplicative, and hence is not a homomorphism unless G is abelian. Thus it is not obvious, and in general it is false, that the linear map $i \otimes \nu\colon B(\mathscr{H}) \odot \mathscr{L}(G) \to B(\mathscr{H}) \otimes \mathscr{L}(G)$ extends to the von Neumann algebra $B(\mathscr{H}) \otimes \mathscr{L}(G)$; even if it does, it is not clear what algebraic properties the map $i \otimes \nu$ will have. (In both [12] and [23], some more detailed arguments might have been helpful.)

We can dodge the factor $\sigma(t, -t)$ by normalising the cocycle σ so that $\sigma(t, -t) = 1$: every cocycle is equivalent to a normalised one. Equivalently, σ is normalised if any σ-representation V satisfies $i \otimes \mu(V) = V^*$. The corresponding property of a dual cocycle would be $i \otimes \nu(V) = V^*$, which we can interpret, without direct reference to $i \otimes \nu$, as

$$\begin{aligned}&(V(h \otimes \bar{\eta}) \mid k \otimes \bar{\xi}) \\ &\quad = (V^*(h \otimes \xi) \mid k \otimes \eta) \qquad \text{for } h, k \in \mathscr{H},\quad \xi, \eta \in L^2(G). \qquad (1.5)\end{aligned}$$

We shall now define ω-corepresentations, and declare ω to be normalised if there is an ω-corepresentation satisfying (1.5). We do not know if every dual cocycle is equivalent to a normalised one, although [23, Theorem 6] says this is true when G is compact.

DEFINITION 1.3. If ω is a dual cocycle on G, then an *ω-corepresentation* of G on $\mathscr{H}$ is a unitary operator $V \in B(\mathscr{H}) \otimes \mathscr{L}(G)$ such that

$$V_{12} V_{13} = i \otimes \delta_G(V)\, \omega_{23}^* \quad \text{in} \quad B(\mathscr{H}) \otimes \mathscr{L}(G) \otimes \mathscr{L}(G); \tag{1.6}$$

when $\omega = 1 \otimes 1$, we say V is a corepresentation.

Remark 1.4. Since $\Sigma \circ \delta_G = \delta_G$, a unitary V in $B(\mathscr{H}) \otimes \mathscr{L}(G)$ is an ω-corepresentation if and only if

$$V_{12}^* V_{13}^* = \Sigma(\omega)_{23}\, i \otimes \delta_G(V^*). \tag{1.7}$$

Again, we look at the case of abelian G for motivation. A $\bar{\sigma}$-representation of $\hat{G}$ is a Borel homomorphism $U: G \to U(\mathscr{H})$ such that

$$U_s U_t = \overline{\sigma(s, t)}\, U_{s+t} \qquad \text{for} \quad s, t \in \hat{G}. \tag{1.8}$$

If we view U as the unitary element $t \mapsto U_t$ of $L^\infty(\hat{G}, B(\mathscr{H})) \cong B(\mathscr{H}) \otimes L^\infty(\hat{G})$, then $B(\mathscr{H}) \otimes L^\infty(\hat{G}) \otimes L^\infty(\hat{G}) \cong L^\infty(\hat{G} \times \hat{G}, B(\mathscr{H}))$, and (1.8) is equivalent to

$$U_{12} U_{13} = \bar{\sigma}_{23}\, i \otimes \alpha_G(U).$$

Since ω is the Fourier transform of σ, the Fourier transform V of U satisfies (1.6). In the abelian case, σ_{23} is central, and it does not matter which side σ_{23} goes on; in general we have to be more careful. We have chosen to put the ω^* on the right of $i \otimes \delta_G(V)$ because we want $\operatorname{Ad} V: T \mapsto V(T \otimes 1) V^*$ to be a coaction of G on $B(\mathscr{H})$:

$$\begin{aligned}
(\operatorname{Ad} V \otimes i) \circ \operatorname{Ad} V(T) &= (\operatorname{Ad} V_{12} V_{13})(T \otimes 1 \otimes 1) \\
&= \operatorname{Ad}(i \otimes \delta_G(V)\, \omega_{23}^*)(T \otimes 1 \otimes 1) \\
&= \operatorname{Ad}(i \otimes \delta_G(V))(T \otimes \delta_G(1)) \\
&= i \otimes \delta_G(\operatorname{Ad} V(T)).
\end{aligned}$$

LEMMA 15. *If V is an ω-corepresentation of G on $\mathscr{H}$, then $\mu(f) = S_f(V)$ defines a norm-decreasing homomorphism of $A(G, \omega)$ into $B(\mathscr{H})$.*

Proof. We have $\|\mu(f)\| \leqslant \|f\|$ because $\|V\|=1$ and $\|S_f\|=\|f\|$, and a calculation shows that μ is multiplicative: if $\phi \in B(\mathscr{H})_*$, then

$$\begin{aligned}\langle \mu(f *_\omega g), \phi\rangle &= \langle V, \phi \otimes (f *_\omega g)\rangle \\ &= \langle i \otimes \delta_G(V)\,\omega_{23}^*, \phi \otimes f \otimes g\rangle \\ &= \langle V_{12} V_{13}, \phi \otimes f \otimes g\rangle \\ &= \langle V(\mu(g) \otimes 1), \phi \otimes f\rangle \\ &= \langle \mu(f)\mu(g), \phi\rangle.\end{aligned}$$

LEMMA 1.6. *If $V = W_G^* \Sigma(\omega^*)$, then V is an ω-corepresentation of G on $L^2(G)$, which we call the* regular ω-corepresentation *of G.*

Proof. We calculate

$$\begin{aligned}V_{12} V_{13} &= (W_G^*)_{12}\, \Sigma(\omega^*)_{12}\, (W_G^*)_{13}\, \Sigma(\omega^*)_{13} \\ &= (W_G^*)_{12}\, (W_G^*)_{13}\, \Sigma \otimes i((W_G)_{23}\, \omega_{12}^* (W_G^*)_{23})\, \Sigma(\omega^*)_{13} \\ &= (W_G^*)_{12}\, (W_G^*)_{13}\, \Sigma \otimes i(i \otimes \delta_G(\omega^*))\, \Sigma(\omega^*)_{13} \\ &= i \otimes \delta_G(W_G^*)(\Sigma \otimes i) \circ (i \otimes \Sigma)(i \otimes \delta_G(\omega^*)\, \omega_{23}^*) \\ &= i \otimes \delta_G(W_G^*)(\Sigma \otimes i) \circ (i \otimes \Sigma)(\delta_G \otimes i(\omega^*)\, \omega_{12}^*).\end{aligned}$$

Since $(\Sigma \otimes i) \circ (i \otimes \Sigma) \circ (\delta_G \otimes i) = (i \otimes \delta_G) \circ \Sigma$, this implies

$$V_{12} V_{13} = i \otimes \delta_G(W_G^*)\; i \otimes \delta_G(\Sigma(\omega^*))\, \omega_{23}^* = i \otimes \delta_G(V)\, \omega_{23}^*,$$

as required.

To see why we call this the regular ω-corepresentation, suppose G is abelian and ω is the Fourier transform of $\sigma \in Z^2(\hat{G}, \mathbf{T})$. A quick calculation shows that $\mathrm{Ad}(\mathscr{F} \otimes \mathscr{F})(W_G)$ is the operator $V_{\hat{G}}$ on $L^2(\hat{G} \times \hat{G})$ defined by $V_{\hat{G}}\xi(s, t) = \xi(s+t, t)$, and hence $\mathrm{Ad}(\mathscr{F} \otimes \mathscr{F})(W_G^* \Sigma\sigma(\omega^*))$ is the operator U defined by

$$U(\xi)(s, t) = \overline{\sigma(s, s-t)}\, \xi(s-t, t).$$

This is multiplication by the operator-valued function $t \mapsto U_t$, where

$$(U_t \xi)(s) = \overline{\sigma(s, s-t)}\, \xi(s-t)$$

is one version of the regular $\bar{\sigma}$-representation of $\hat{G}$ on $L^2(\hat{G})$.

DEFINITION 1.7. A dual cocycle ω on G is *normalised* if there is an ω-corepresentation U of G on a Hilbert space $\mathscr{H}$ satisfying

$$(U(h \otimes \bar{\eta}) \mid k \otimes \bar{\xi}) = (U^*(h \otimes \xi) \mid k \otimes \eta) \tag{1.9}$$

for $h, k \in \mathscr{H}$, $\xi, \eta \in L^2(G)$.

PROPOSITION 1.8. *Suppose ω is a normalised dual cocycle on G. Then for any ω-corepresentation V of G on a Hilbert space $\mathscr{H}_V$, we have*

$$(V(h\otimes\bar{\eta})\,|\,k\otimes\bar{\xi})=(V^*(h\otimes\xi)\,|\,k\otimes\eta) \tag{1.10}$$

for $h, k\in\mathscr{H}_V$ and $\xi, \eta\in L^2(G)$.

We begin by observing that we can stick in an extra variable, which will give us room to manoeuvre; since we shall want to do this repeatedly, we state it as a separate lemma.

LEMMA 1.9. *Let V be an ω-corepresentation of G. Then V satisfies* (1.10) *if and only if*

$$(V_{12}(h\otimes\bar{\beta})\,|\,k\otimes\bar{\alpha})=(V_{12}^*(h\otimes\alpha)\,|\,k\otimes\beta) \tag{1.11}$$

for all $h, k\in\mathscr{H}_V$ and $\alpha, \beta\in L^2(G\times G)$.

Proof. Taking $\alpha=\xi\otimes f$, $\beta=\eta\otimes g$ in (1.11) gives

$$(V(h\otimes\bar{\eta})\,|\,k\otimes\bar{\xi})(\bar{g}\,|\,\bar{f})=(V^*(h\otimes\xi)\,|\,k\otimes\eta)(f\,|\,g),$$

which reduces to (1.10) (recall that we always have $(\bar{g}\,|\,\bar{f})=(f\,|\,g)$ in an L^2-space). This observation also shows that (1.10) implies (1.11) for elementary tensors α, β, and the full generality of (1.11) follows, since both sides are linear in α, antilinear in β, and continuous.

The next lemma is the analogue of the well-known absorbing property of the regular representation L: $L\otimes U\sim L\otimes 1$ for any unitary representation U (cf. [7, XI.13.11; 23, Lemma 11]). We shall use it to show that both (1.9) and (1.10) are equivalent to the analogous property of the regular ω-representation $W_G^*\Sigma(\omega^*)$.

LEMMA 1.10. *Suppose ω is a dual cocycle, V is an ω-corepresentation of G, and W is an ordinary corepresentation of G. Then $U=W_{23}V_{13}$ is an ω-corepresentation on $\mathscr{H}_V\otimes\mathscr{H}_W$. When $W=W_G^*$, U is equivalent to $1\otimes W_G^*\Sigma(\omega^*)$: indeed,*

$$\mathrm{Ad}(V^*\otimes 1)((W_G^*)_{23}V_{13})=(W_G^*\Sigma(\omega^*))_{23}.$$

Proof. First,

$$\begin{aligned} U_{123}\, U_{124} &= W_{23}\, V_{13}\, W_{24}\, V_{14} = W_{23}\, W_{24}\, V_{13}\, V_{14} \\ &= i \otimes \delta_G(W)_{234}\, i \otimes \delta_G(V)_{134}\, \omega_{34}^* \\ &= i \otimes i \otimes \delta_G(W_{23}\, V_{13})\, \omega_{34}^*, \end{aligned}$$

so U is an ω-corepresentation. Next, using (1.7), we have

$$V_{12}^*(W_G^*)_{23}\, V_{13} = (W_G^*)_{23}\, i \otimes \delta_G(V^*)\, V_{13} = (W_G^*)_{23}\, \Sigma(\omega^*)_{23}\, V_{12}^*,$$

which completes the proof.

Proof of Proposition 1.8. We claim that V satisfies (1.11) if and only if $X = (W_G^*)_{23}\, V_{13}$ satisfies the appropriate version of (1.10) (i.e., Eq. (1.12) below). First, suppose V satisfies (1.11), and let $h, k \in \mathscr{H}$, $\gamma, \delta, \xi, \eta \in L^2(G)$. If we view the middle copy of $L^2(G)$ as the extra one in (1.11), Lemma 1.9 gives

$$\begin{aligned} (X^*(h \otimes \gamma \otimes \xi) \mid k \otimes \delta \otimes \eta) &= (V_{13}^*(W_G)_{23}\, (h \otimes \gamma \otimes \xi) \mid k \otimes \delta \otimes \eta) \\ &= (V_{13}(h \otimes \overline{\delta \otimes \eta}) \mid k \otimes \overline{W_G(\gamma \otimes \xi)}). \end{aligned}$$

We trivially have $\overline{W_G(\gamma \otimes \xi)} = W_G(\bar{\gamma} \otimes \bar{\xi})$, so this is equal to

$$((W_G^*)_{23}\, V_{13}(h \otimes \bar{\delta} \otimes \bar{\eta}) \mid k \otimes \bar{\gamma} \otimes \bar{\xi}).$$

In the middle variable, $(W_G^*)_{23}\, V_{13}$ acts as a multiplication operator, and from the equation $(M_f\, \bar{\delta} \mid \bar{\gamma}) = (M_f \gamma \mid \delta)$ we deduce that

$$\begin{aligned} (X^*(h \otimes \gamma \otimes \xi) \mid k \otimes \delta \otimes \eta) &= ((W_G^*)_{23}\, V_{13}(h \otimes \gamma \otimes \bar{\eta}) \mid k \otimes \delta \otimes \bar{\xi}) \\ &= (X(h \otimes \gamma \otimes \bar{\eta}) \mid k \otimes \delta \otimes \bar{\xi}). \end{aligned}$$

Again, we use linearity and continuity to extend this equation from elementary tensors, and obtain

$$(X^*(\alpha \otimes \xi) \mid \beta \otimes \eta) = (X(\alpha \otimes \bar{\eta}) \mid \beta \otimes \bar{\xi}) \qquad \text{for} \quad \alpha, \beta \in \mathscr{H} \otimes L^2(G). \tag{1.12}$$

On the other hand, if X satisfies (1.12), we can take $\alpha = h \otimes \gamma$, $\beta = k \otimes \delta$, and apply the above reasoning to the left-hand side. We obtain

$$(V_{13}(h \otimes \overline{\delta \otimes \eta}) \mid k \otimes \overline{W_G(\gamma \otimes \xi)}) = (V_{13}^*(h \otimes W_G(\gamma \otimes \xi)) \mid k \otimes \delta \otimes \eta).$$

As usual, this extends from elementary tensors $\gamma \otimes \xi$, $\delta \otimes \eta$ to arbitrary functions in $L^2(G) \otimes L^2(G) = L^2(G \times G)$, and (1.11) follows since W_G is surjective.

By Lemma 1.10, the unitary operator V on $\mathscr{H}\otimes L^2(G)$ satisfies $\mathrm{Ad}(V^*\otimes 1)(X)=(W_G^*\Sigma(w^*))_{23}$. If we now replace α, β in (1.12) by $V(h\otimes\gamma)$, $V(k\otimes\delta)$, and suppress the first variable, we find that (1.12) is equivalent to

$$(W_G^*\Sigma(\omega^*)(\gamma\otimes\bar{\eta})\,|\,\delta\otimes\bar{\xi})=(\Sigma(\omega)W_G(\gamma\otimes\xi)\,|\,\delta\otimes\eta)\quad \text{for } \gamma,\delta,\xi,\eta\in L^2(G).$$

Since this last statement is independent of the ω-corepresentation V we started with, this proves the proposition.

LEMMA 1.11. *Let* ω *be a normalised dual cocycle on* G. *Then* $\nu\otimes\nu(\omega^*)=\Sigma(\omega)$.

Proof. (We would just apply $i\otimes\nu\otimes\nu$ to (1.6) if it made sense to do so.) Let U be an ω-corepresentation satisfying (1.9). Then two applications of (1.9) give

$$\begin{aligned}(U_{12}^*U_{13}^*(h\otimes\alpha\otimes\xi)\,|\,k\otimes\beta\otimes\eta)&=(U_{12}U_{13}(h\otimes\bar{\beta}\otimes\bar{\eta})\,|\,k\otimes\bar{\alpha}\otimes\bar{\xi})\\&=(i\otimes\delta_G(U)\,\omega_{23}^*(h\otimes\bar{\beta}\otimes\bar{\eta})\,|\,k\otimes\bar{\alpha}\otimes\bar{\xi}).\end{aligned}$$

Now

$$\begin{aligned}(\omega^*(\bar{\beta}\otimes\bar{\eta})\,|\,\delta\otimes\gamma)&=(\nu\otimes\nu(\omega^*)(\bar{\delta}\otimes\bar{\gamma})\,|\,\beta\otimes\eta)\\&=(\bar{\delta}\otimes\bar{\gamma}\,|\,\nu\otimes\nu(\omega)(\beta\otimes\eta))\\&=(\overline{\nu\otimes\nu(\omega)(\beta\otimes\eta)}\,|\,\delta\otimes\gamma),\end{aligned}$$

and hence

$$\begin{aligned}&(U_{12}^*U_{13}^*(h\otimes\alpha\otimes\xi)\,|\,k\otimes\beta\otimes\eta)\\&\quad=(i\otimes\delta_G(U)\,h\otimes\overline{\nu\otimes\nu(\omega)(\beta\otimes\eta)}\,|\,k\otimes\overline{\alpha\otimes\xi})\\&\quad=(U_{12}(h\otimes W_G^*(\overline{\nu\otimes\nu(\omega)(\beta\otimes\eta)}))\,|\,k\otimes W_G^*(\overline{\alpha\otimes\xi}))\\&\quad=(U_{12}(h\otimes\overline{W_G^*(\nu\otimes\nu(\omega)(\beta\otimes\eta))})\,|\,k\otimes\overline{W_G^*(\alpha\otimes\xi)})\\&\qquad\qquad\qquad\qquad\text{since }\ \overline{W_G^*\xi}=W_G^*\bar{\xi}\\&\quad=(U_{12}^*(h\otimes W_G^*(\alpha\otimes\xi))\,|\,k\otimes W_G^*(\nu\otimes\nu(\omega)(\beta\otimes\eta)))\qquad\text{by Lemma 1.9}\\&\quad=(i\otimes\delta_G(U^*)\,h\otimes\alpha\otimes\xi\,|\,k\otimes(\nu\otimes\nu)(\omega)(\beta\otimes\eta)).\end{aligned}$$

Thus we have $U_{12}^*U_{13}^*=\nu\otimes\nu(\omega^*)_{23}\ i\otimes\delta_G(U^*)$, and the result follows from Remark 1.4.

Remark 1.12. For compact G, this is a special case of [23, Lemma 13]; our proof avoids direct manipulation of $i \otimes \nu$. The lemma implies that condition (3.4) in the definition of normalised cocycle used in [12] is redundant. We point out that, even in the abelian case, the equation $\nu \otimes \nu(\omega) = \Sigma(\omega^*)$ is distinctly weaker than normalisation. For example, take $\hat{G} = \mathbf{Z}_2 = \{\pm 1\}$, define $\rho : \hat{G} \to \mathbf{T}$ by $\rho(1) = 1$, $\rho(-1) = i$, and let $\sigma = \partial\rho$. Then $\mu \otimes \mu(\sigma)(s, t) = \sigma(s^{-1}, t^{-1}) = \sigma(s, t) = -1$ if $s = t = -1$; it is equal to 1 otherwise. But

$$\sigma((-1)^{-1}, (-1)) = \sigma(-1, -1) \neq 1.$$

PROPOSITION 1.13. *If ω is a normalised dual cocycle, then $A(G, \omega)$ is a Banach $*$-algebra with involution given by $f^*(t) = \overline{f(t)}$.*

Proof. Since this is the normal involution on $A(G)$, and we have not changed the Banach space structure, the only thing we have to check is $(f *_\omega g)^* = g^* *_\omega f^*$. Recall that the involution on $A(G)$ can be described in terms of ν by $\langle T, f^* \rangle = \overline{\langle \nu(T^*), f \rangle}$. Thus for $T \in \mathscr{L}(G)$, we have

$$\begin{aligned}
\langle T, (f *_\omega g)^* \rangle &= \overline{\langle \nu(T^*), f *_\omega g \rangle} = \overline{\langle \delta_G(\nu(T^*))\, \omega^*, f \otimes g \rangle} \\
&= \overline{\langle \delta_G(\nu(T^*))\, \Sigma(\nu \otimes \nu(\omega)), f \otimes g \rangle} && \text{by Lemma 1.11} \\
&= \overline{\langle \Sigma(\delta_G(\nu(T^*))\, \nu \otimes \nu(\omega)), f \otimes g \rangle} && \text{since } \Sigma \circ \delta_G = \delta_G \\
&= \overline{\langle \delta_G(\nu(T^*))(\nu \otimes \nu(\omega)), g \otimes f \rangle} \\
&= \overline{\langle \nu \otimes \nu(\omega\, \delta_G(T^*)), g \otimes f \rangle} && \text{since } \delta_G \circ \nu = (\nu \otimes \nu) \circ \delta_G \\
&= \langle (\omega\, \delta_G(T^*))^*, g^* \otimes f^* \rangle \\
&= \langle T, g^* *_\omega f^* \rangle,
\end{aligned}$$

as required.

THEOREM 1.14. *Suppose ω is a normalised dual cocycle and V is an ω-corepresentation of G on $\mathscr{H}$. Then there is a norm-decreasing, non-degenerate $*$-representation μ of $A(G, \omega)$ on $\mathscr{H}$ such that $\mu(f) = S_f(V)$ for $f \in A(G)$.*

Proof. We have already seen that μ is multiplicative and norm-decreasing in Lemma 1.5. Since

$$(\mu(f)\, h \mid k) = 0 \qquad \text{for all} \quad h \in \mathscr{H}, \quad f \in A(G)$$

if and only if

$$(V(h\otimes\xi)\,|\,k\otimes\eta)=0 \qquad \text{for all} \quad h\in\mathscr{H} \quad \text{and} \quad \xi,\eta\in L^2(G),$$

the nondegeneracy of μ follows from the surjectivity of V. It remains to show that $\mu(f^*)=\mu(f)^*$. Suppose f is the functional $T\mapsto(T\xi\,|\,\eta)$ in $\mathscr{L}(G)_*$, so that the conjugate f^* is the functional $T\mapsto(T\bar{\xi}\,|\,\bar{\eta})$. Then

$$\begin{aligned}(\mu(f^*)\,h\,|\,k) &= (V(h\otimes\bar{\xi})\,|\,k\otimes\bar{\eta})\\ &= (V^*(h\otimes\eta)\,|\,k\otimes\xi) \qquad \text{by Proposition 1.8}\\ &= \overline{(V(k\otimes\xi)\,|\,h\otimes\eta)}=\overline{(S_f(V)\,k\,|\,h)}\\ &= (h\,|\,S_f(V)\,k)=(h\,|\,\mu(f)\,k),\end{aligned}$$

as required.

Problem. We would very much like to know whether *every* *-representation of $A(G,\omega)$ is given by slicing an ω-corepresentation. This is true if $\omega=1\otimes 1$ by [14, Theorem A1], but the proof given there depends crucially on the commutativity of $A(G)$. In general, one can show that every *-representation μ has the form $\mu(f)=S_f(V)$ for some operator V satisfying (1.6) and (1.10): the problem is to show that V has to be unitary.

DEFINITION 1.15. Let ω be a normalised dual cocycle on a locally compact group G, and μ be the *-representation of $A(G,\omega)$ given by $\mu(f)=S_f(W_G^*\Sigma(\omega^*))$. The *(reduced) twisted dual-group algebra* $C_r^*(\hat{G},\omega)$ is the closure of $\mu(A(G,\omega))$ in $B(L^2(G))$.

LEMMA 1.16. *When G is a second countable abelian group, and the normalised dual cocycle ω is the Fourier transform of a normalised cocycle σ on $\hat{G}$, the Fourier transform $\mathscr{F}h(x)=\int\langle x,s\rangle\, h(s)\,ds$ of $L^1(\hat{G})$ onto $A(G)$ induces an isomorphism of the twisted group algebra $C^*(\hat{G},\bar{\sigma})$ onto $C_r^*(\hat{G},\omega)$.*

Proof. We have already seen that $\mathscr{F}$ is a *-isomorphism of $L^1(\hat{G},\bar{\sigma})$ onto $A(G,\omega)$, so we just have to check that the C^*-norms match up. Moving the representation $\mu\circ\mathscr{F}$ over to $L^2(\hat{G})$ via the Fourier transform $\mathscr{F}: L^2(\hat{G})\to L^2(G)$ gives the representation

$$\begin{aligned}\pi(f) &= \mathscr{F}^{-1}\mu(\mathscr{F}f)\,\mathscr{F}=\mathscr{F}^{-1}(S_{\mathscr{F}f}(W_G^*\Sigma(\omega^*)))\,\mathscr{F}\\ &= S_f((\mathscr{F}^{-1}\otimes\mathscr{F}^{-1})\,W_G^*\Sigma(\omega^*)(\mathscr{F}\otimes\mathscr{F}))\\ &= S_f((\mathscr{F}^{-1}\otimes\mathscr{F}^{-1})\,W_G^*(\mathscr{F}\otimes\mathscr{F})\,\Sigma(\bar{\sigma})).\end{aligned}$$

Since $(\mathscr{F}^{-1}\otimes\mathscr{F}^{-1})\,W_G^*(\mathscr{F}\otimes\mathscr{F})$ is the operator V_G^* given by $V_G^*h(s,t)=h(s-t,t)$, and every $f\in L^1(\hat{G})$ is the pointwise product $\xi\bar{\eta}$ of two functions in $L^2(\hat{G})$, we can compute

$$\begin{aligned}(\pi(f)\,h\,|\,k) &= (V_G^*\Sigma(\bar{\sigma})(h\otimes\xi)\,|\,k\otimes\eta)\\ &= \iint \overline{\sigma(t,s-t)}\,h(s-t)\,\xi(t)\,\overline{k(s)\,\eta(t)}\,ds\,dt\\ &= \int\left(\int f(t)\,\overline{\sigma(t,s-t)}\,h(s-t)\,dt\right)\overline{k(s)}\,ds.\end{aligned}$$

Thus π is the integrated form of the representation $U\colon \hat{G}\to U(L^2(\hat{G}))$ defined by

$$(U_t h)(s)=\overline{\sigma(t,s-t)}\,h(s-t).$$

This is one realisation of the regular $\bar{\sigma}$-representation of $\hat{G}$, and its integrated form is faithful on $C^*(\hat{G},\bar{\sigma})$ because $\hat{G}$ is amenable (e.g., [16, Theorem 3.11]). (This is where we use the second countability of G; we lack a reference for the general case.) Thus π is isometric, and since μ is isometric by definition, $\mathscr{F}$ extends to an isometric *-isomorphism of $C^*(\hat{C},\bar{\sigma})$ onto $C_r^*(\hat{G},\omega)$.

Problem. There is considerable evidence that the dual-group algebra $C_0(G)$ is amenable in an appropriate sense (e.g., [5, 19]), and hence we conjecture that the regular representation always induces an isomorphism of the C^*-enveloping algebra $C^*(A(G,\omega))$ onto $C_r^*(\hat{G},\omega)$.

PROPOSITION 1.17. *Suppose ω is a normalised dual cocycle on a locally compact group G. For each $y\in G$, $\alpha_y(f)(x)=f(y^{-1}x)$ defines a *-automorphism α_y of $A(G,\omega)$, and α is then a strongly continuous action of G on $A(G,\omega)$. If μ is the representation of $A(G,\omega)$ obtained by slicing the regular ω-corepresentation $W_G^*\Sigma(\omega^*)$, then (μ, R^G) is a covariant representation of $(A(G,\omega),G,\alpha)$. In particular, it follows that α extends to a strongly continuous action of G on $C_r^*(\hat{G},\omega)$.*

Proof. We first observe that

$$\langle L_{y^{-1}}T,f\rangle=\langle T,\alpha_y(f)\rangle \qquad \text{for} \quad T\in\mathscr{L}(G); \tag{1.13}$$

to see this, choose $\xi,\ \eta\in L^2(G)$ such that $\langle T,f\rangle=(T\xi\,|\,\eta)$, and expand both sides for $T=L_x$, $x\in G$. From (1.13) we easily verify that $\alpha_y(f *_\omega g)=\alpha_y(f) *_\omega \alpha_y(g)$; since we trivially have $\alpha_y(f^*)=\alpha_y(f^*)$, it follows that α_y

is a *-automorphism of $A(G, \omega)$. If again $f \in A(G) = \mathscr{L}(G)_*$ is given by $T \mapsto (T\xi \mid \eta)$, we also have

$$|\langle T, \alpha_y(f) - f\rangle| = |((L_{y^{-1}} T - T)\xi \mid \eta)| \leqslant \|T\| \, \|\xi\| \, \|L_y \eta - \eta\|,$$

and hence $\|\alpha_y(f) - f\|_{A(G)} \to 0$ as $y \to e$. Thus α is strongly continuous.

For the covariance, we compute,

$$\begin{aligned} R_y \mu(f) R_y^* &= R_y S_f(W_G^* \Sigma(\omega^*)) R_y^* \\ &= S_f((R_y \otimes 1) W_G^* \Sigma(\omega^*)(R_y^* \otimes 1)) \\ &= S_f((R_y \otimes 1) W_G^*(R_y^* \otimes 1) \Sigma(\omega^*)), \end{aligned}$$

because R_y is in $\mathscr{R}(G) = \mathscr{L}(G)'$. A straightforward calculation shows that

$$(R_y \otimes 1) W_G^*(R_y^* \otimes 1) = (1 \otimes L_y^*) W_G^*,$$

and (1.13) implies that $S_f((1 \otimes L_y^*)T) = S_{\alpha_y(f)}(T)$, so we have

$$R_y \mu(f) R_y^* = S_f((1 \otimes L_y^*) W_G^* \Sigma(\omega^*)) = S_{\alpha_y(f)}(W_G^* \Sigma(\omega^*)) = \mu(\alpha_y(f)).$$

It follows immediately that α_y is isometric for the C^*-norm, and hence extends to a *-automorphism of $C_r^*(\hat{G}, \omega)$. Since $\mu: A(G, \omega) \to B(L^2(G))$ is norm-decreasing, the continuity of α on $A(G, \omega)$ implies that $\alpha: G \to \operatorname{Aut} C_r^*(\hat{G}, \omega)$ is strongly continuous.

2. Induced C^*-Algebras and Induced Representations

Suppose H is a closed subgroup of a locally compact group G and $\beta: H \to \operatorname{Aut} B$ is an action of H on a C^*-algebra B. Then the *induced C^*-algebra* $\operatorname{Ind}_H^G(B, \beta)$ is the subalgebra

$$\left\{ f \in C_b(G, B) \;\middle|\; \begin{array}{l} f(xh) = \beta_{h^{-1}}(f(x)) \text{ for } x \in G,\ h \in H, \\ \text{and } xH \mapsto \|f(x)\| \text{ is in } C_0(G/H) \end{array} \right\}$$

of $C_b(G, B)$. The larger group G acts by left translation on $\operatorname{Ind}_H^G B$: $\tau_x(f)(y) = f(x^{-1}y)$. We refer to $(\operatorname{Ind}_H^G B, G, \tau)$ as the *induced system*. These systems have been studied in, for example, [15, 20]. We can easily construct elements of Ind B:

LEMMA 2.1. *Let $\beta: H \to \operatorname{Aut} B$, and for $f \in C_c(G)$, $b \in B$ define*

$$(f \odot b)(x) = \int_H f(xh)\, \beta_h(b)\, dh.$$

Then the functions $f \odot b$ span a dense subspace of $\operatorname{Ind}_H^G(B, \beta)$*—indeed, f and b need only run through dense subspaces of $C_c(G)$ and B.*

Proof. A routine calculation shows that these functions are in $\operatorname{Ind}_H^G B$, and then a partition of unity argument (on G/H) shows that their span is dense.

We seek to construct covariant representations of $(\operatorname{Ind}_H^G B, G, \tau)$ from covariant representations of (B, H, β). It turns out that, if (π, S) is a covariant representation of (B, H, β) on $\mathscr{H}$, then $(\operatorname{Ind}_H^G B, G, \tau)$ acts naturally in the Hilbert space of the induced representation $\operatorname{Ind}_H^G S$. To see this, it seems most convenient to use Blattner's formulation of $\operatorname{Ind}_H^G S$, which we briefly review (cf. [7, XI.10.8]).

Suppose S is a unitary representation of H on a Hilbert space $\mathscr{H} = \mathscr{H}_S$. We let Δ and δ denote the modular functions on G and H, and fix a continuous rho-function $\rho: G \to (0, \infty)$, so that

$$\rho(xh) = \frac{\delta(h)}{\Delta(h)}\, \rho(x) \qquad \text{for} \quad x \in G, \quad h \in H$$

(cf. [7, III.13.2 and III.14.5]). Then by [7, III.13.10], ρ determines a regular Borel measure $\rho^{\#}$ on G/H such that

$$\int_{G/H} \int_H \phi(xh)\, dh\, d\rho^{\#}(xH) = \int_G \rho(x)\, \phi(x)\, dx \qquad \text{for} \quad \phi \in C_c(G).$$

The space $\mathscr{H}_{\operatorname{Ind} S}$ then consists of the locally measurable functions $\phi: G \to \mathscr{H}_S$ such that

$$\phi(xh) = \delta(h)^{1/2}\, \Delta(h)^{-1/2}\, S_{h^{-1}}\, \phi(x) \qquad \text{for} \quad x \in G, \quad h \in H,$$

and such that the function $xH \mapsto \rho(x)^{-1}\, \|\phi(x)\|^2$ is in $L^1(G/H, d\rho^{\#})$; as usual, we identify functions which are equal locally almost everywhere, and then $\mathscr{H}_{\operatorname{Ind} S}$ is a Hilbert space with

$$(\phi \mid \psi) = \int_{G/H} \rho(x)^{-1}\, (\phi(x) \mid \psi(x))\, d\rho^{\#}(xH).$$

The *induced representation* $\operatorname{Ind}_H^G S$ is then defined by

$$(\operatorname{Ind}_H^G S)_y\, (\phi)(x) = \phi(y^{-1}x) \qquad \text{for} \quad \phi \in \mathscr{H}_{\operatorname{Ind} S} \quad \text{and} \quad x, y \in G.$$

LEMMA 2.2. *If $\psi \in C_c(G)$ and $\xi \in \mathscr{H}_S$, then*

$$(\psi \odot \xi)(x) = \int_H \psi(xh)\, \delta(h)^{-1/2}\, \Delta(h)^{1/2}\, S_h(\xi)\, dh$$

defines an element of $\mathscr{H}_{\operatorname{Ind} S}$, and the elements of this form span a dense subspace of $\mathscr{H}_{\operatorname{Ind} S}$.

Proof. Since $\psi \odot \xi$ is continuous, it is certainly measurable, and a straightforward change-of-variables argument shows that it transforms correctly on cosets. To see that $\rho(x)^{-1} \|\psi \odot \xi(x)\|^2$ is integrable, use a change of variables and Fubini's theorem to compute

$$\int_{G/H} \rho(x)^{-1} \|(\psi \odot \xi)(x)\|^2 \, d\rho^\#(xH)$$

$$= \int_H \delta(l)^{-1/2} \Delta(l)^{1/2} (\phi * \bar{\psi})(l^{-1})(S_l \xi \mid \xi) \, dl,$$

where $\phi(y) = \psi(y^{-1})$, and observe that $\phi * \bar{\psi} \in C_c(G)$. To see that these elements span a dense subspace, verify that

$$(\psi \odot \xi \mid \phi) = \int_G \psi(x)(\xi \mid \phi(x)) \, dx,$$

so that no element ϕ of $\mathcal{H}_{\operatorname{Ind} S}$ can be orthogonal to $\operatorname{sp}\{\psi \odot \xi\}$.

PROPOSITION 2.3. *Suppose (π, S) is a covariant representation of (B, H, β). Then there is a representation $\tilde{\pi}$ of $\operatorname{Ind}_H^G(B, \beta)$ on $\mathcal{H}_{\operatorname{Ind} S}$ such that*

$$(\tilde{\pi}(f)\,\phi)(x) = \pi(f(x))\,\phi(x),$$

and $(\tilde{\pi}, \operatorname{Ind}_H^G S)$ is then a covariant representation of $(\operatorname{Ind}_H^G B, G, \tau)$. The representation $\tilde{\pi}$ is faithful if and only if π is.

Proof. We first need to check that $\tilde{\pi}(f)\,\phi$ belongs to $\mathcal{H}_{\operatorname{Ind} S}$. Using the characterisation of locally λ-measurable functions as the λ-a.e. limit of simple functions, and the continuity of f, one can verify that $\tilde{\pi}(f)\,\phi$ is locally measurable. It is routine to check that $\tilde{\pi}(f)\,\phi$ transforms correctly on cosets, and the estimate

$$\int \rho(x)^{-1} \|\tilde{\pi}(f)\,\phi(x)\|^2 \, d\rho^\#(xH) \leqslant \|f\|_\infty^2 \int \rho(x)^{-1} \|\phi(x)\|^2 \, d\rho^\#(xH)$$

implies both that $\tilde{\pi}(f)\,\phi$ is in $\mathcal{H}_{\operatorname{Ind} S}$ and that $\tilde{\pi}(f)$ is a bounded operator. We deduce that $\tilde{\pi}$ is a representation of $\operatorname{Ind} B$ on $\mathcal{H}_{\operatorname{Ind} S}$. Now a trivial calculation shows that $(\tilde{\pi}, \operatorname{Ind} S)$ is a covariant representation of $(\operatorname{Ind} B, G, \tau)$.

Next suppose π is faithful and $\tilde{\pi}(f) = 0$, and fix $\xi \in \mathcal{H}_S$. Then $\tilde{\pi}(f)(\psi \odot \xi) = 0$ says

$$\int_H \psi(xh)\,\delta(h)^{-1/2} \Delta(h)^{1/2} \pi(f(x))(S_h \xi) \, dh = 0 \qquad x\text{—a.e.}$$

Since the integrand is continuous in x and h, it must vanish identically for all x, and by varying $\psi \in C_c(G)$ we can deduce that $\pi(f(x))\,\xi = 0$ for all x, and $f \equiv 0$. Conversely, suppose that $\tilde{\pi}$ is faithful and $b \in B$ is non-zero. If f has support in a neighborhood of the identity in which $\beta_h(b) \sim b$, then $f \odot b(e) \sim b$, and $\tilde{\pi}(f \odot b) \neq 0$. But then there exists x such that

$$0 \neq \pi(f \odot b(x)) = \int f(xh)\, \pi(\beta_h(b))\, dh = \int f(xh)\, S_h \pi(b) S_h^*\, dh,$$

and hence $\pi(b) \neq 0$.

It is well-known that inducing the left regular representation L^H of H to G gives the left regular representation L^G of G (e.g., [7, XI.12.18]), and of course the same is true for the right regular representations R^H, R^G. But we have chosen our conventions to match left rather than right (we are using $\mathscr{L}(G)$, for example), so the explicit operator intertwining R^G and Ind R^H involves the modular function.

LEMMA 2.4. *The operator* $Q\colon L^2(G) \to \mathscr{H}_{\mathrm{Ind}\, R^H}$ *defined by*

$$(Q_\phi(x))(h) = \Delta(x)^{-1/2}\, \phi(hx^{-1})$$

is unitary and satisfies $QR^G Q^* = \mathrm{Ind}\, R^H$.

Proof. Two routine calculations show that for $\phi \in L^2(G)$, we have

$$(Q\phi)(xh)(k) = \delta(h)^{1/2}\, \Delta(h)^{-1/2}\, R^H_{h^{-1}}(Q\phi(x))(k),$$

and that $\|Q\phi\|^2 = \|\phi\|^2$: indeed, for the latter, we have

$$\begin{aligned}
(Q\phi \mid Q\phi) &= \int_{G/H} \int_H \rho(x)^{-1}\, \Delta(x)^{-1}\, |\phi(hx^{-1})|^2\, dh\, d\rho^\#(xH) \\
&= \int_{G/H} \int_H \rho(x)^{-1}\, \Delta(x)^{-1}\, \delta(h)^{-1}\, |\phi(h^{-1}x^{-1})|^2\, dh\, d\rho^\#(xH) \\
&= \int_{G/H} \int_H \rho(xh)^{-1}\, \Delta(xh)^{-1}\, |\phi(h^{-1}x^{-1})|^2\, dh\, d\rho^\#(xH) \\
&= \int_G \Delta(x)^{-1}\, |\phi(x^{-1})|^2\, dx = \|\phi\|^2,
\end{aligned}$$

which shows that $Q\phi \in \mathscr{H}_{\mathrm{Ind}\, R^H}$, that Q is isometric, and, by reversing the calculation, that for any measurable function ϕ on G,

$$\|\phi\|^2 = \int_{G/H} \int_H \rho(x)^{-1}\, \Delta(x)^{-1}\, |\phi(hx^{-1})|^2\, dh\, d\rho^\#(xH). \tag{2.1}$$

Thus to see that Q is unitary, we need to show it has dense range: we show that if $\psi \in C_c(G)$ and $\xi \in C_c(H) \subset L^2(H)$, then $\psi \odot \xi$ belongs to the range of Q. For we can define a continuous function ϕ on G by $\phi(x) = \Delta(x)^{-1/2}(\psi \odot \xi)(x^{-1})(e)$, and then

$$\Delta(x)^{-1/2}\phi(hx^{-1}) = (\psi \odot \xi)(x)(h). \tag{2.2}$$

Plugging this into (2.1) shows that $\|\phi\|_2^2 = \|\psi \odot \xi\|^2$, so that $\phi \in L^2(G)$, and (2.2) says $Q\phi = \psi \odot \xi$. Since $\{\psi \odot \xi\}$ is dense in $\mathscr{H}_{\operatorname{Ind} R^H}$ by Lemma 2.2, we deduce that Q is unitary, as claimed. Now a final calculation shows that $QR_x^G = (\operatorname{Ind} R^H)_x Q$.

3. The Twisted Dual-Group Algebras of Inflated Cocycles

One way to construct cocycles on a group G is to start with a cocycle on a quotient G/N, and inflate it to a cocycle on G by composing with the quotient map $q \times q$ of $G \times G$ onto $G/N \times G/N$. Correspondingly, we can construct dual cocycles on G from ones on a subgroup H. For this, we shall need to know that $\mathscr{L}(H)$ embeds in $\mathscr{L}(G)$, just as $L^\infty(G/N)$ embeds via q^* in $L^\infty(G)$; after we have this, it is quite easy to inflate dual cocycles (Proposition 3.2). We shall then prove our main theorem.

Lemma 3.1. (Herz [9]). *Suppose G is a second countable locally compact group and H a closed subgroup. Then there is a unique normal $*$-monomorphism I of $\mathscr{L}(H)$ into $\mathscr{L}(G)$ such that $I(L_h^H) = L_h^G$ for $h \in H$. Indeed, if $c\colon H\backslash G \to G$ is a Borel section, there is a unitary operator V on $L^2(G \times H)$ such that*

$$V\xi(s, h) = \xi(h^{-1}c(Hs), sc(Hs)^{-1}h),$$

and then I satisfies

$$V(1 \otimes T)V^* = I(T) \otimes 1 \qquad \text{for} \quad T \in \mathscr{L}(H). \tag{3.1}$$

Dually, restricting functions to H is a norm-decreasing homomorphism of $A(G) = \mathscr{L}(G)_*$ onto $A(H) = \mathscr{L}(H)_*$ such that

$$\langle I(T), f\rangle = \langle T, f|_H\rangle \qquad \text{for} \quad T \in \mathscr{L}(H), \quad f \in A(G).$$

Proof. (This is basically Herz' proof.) We first note that $I \otimes L^H$, $L^G \otimes I$ extend to faithful normal representations of $\mathscr{L}(H)$ and $\mathscr{L}(G)$ on

$L^2(G\times H)$, so (3.1) will uniquely determine a normal homomorphism I with the required properties if

$$V(1\otimes L_h^H)V^* = L_h^G\otimes 1 \qquad \text{for} \quad h\in H.$$

But this is an easy calculation, and the first part follows.

For the second part, we just need to observe that for $f\in A(G)$, the functional $T\mapsto \langle I(T), f\rangle$ is weak* continuous and hence lies in $\mathscr{L}(H)_*$. Since $\langle I(L_h^H), f\rangle = \langle L_h^G, f\rangle = f(h)$, the element of $A(H)$ corresponding to this functional is $f|_H$, and since I is norm-preserving, we have

$$\|f|_H\|_{A(H)} = \|f|_H\|_{\mathscr{L}(H)_*} \leqslant \|f\|_{\mathscr{L}(G)_*} = \|f\|_{A(G)}.$$

The surjectivity of $f\to f|_H$ follows from the Hahn–Banach theorem because I embeds $\mathscr{L}(H)$ isometrically as a weak* closed subspace of $\mathscr{L}(G)$.

PROPOSITION 3.2. *Let H be a closed subgroup of a second countable group G, and suppose ω is a dual cocycle on H. Then $I\otimes I(\omega)$ is a dual cocycle on G. If ω is normalised, so is $I\otimes I(\omega)$.*

Proof. For the first part, we just need to observe that because both $\delta_G\circ I$ and $(I\otimes I)\circ\delta_H$ are normal extensions of the map $L_h^H\to L_h^G\otimes L_h^G$, they agree on $\mathscr{L}(H)$, and thus the cocycle identity

$$I\otimes I(\omega)_{12}\,\delta_G\otimes i(I\otimes I(\omega)) = I\otimes I(\omega)_{23}\, i\otimes\delta_G(I\otimes I(\omega))$$

follows easily from the corresponding property of ω. Now suppose that W is an ω-corepresentation of H on $\mathscr{H}$. We claim that $U = i\otimes I(W)$ is then an $I\otimes I(\omega)$-corepresentation of G on $\mathscr{H}$ satisfying (1.9). First of all, the identity

$$U_{12}U_{13} = i\otimes\delta_G(U)\, I\otimes I(\omega)_{23}^*$$

follows easily from the corresponding property of W and the equation $\delta_G\circ I = (I\otimes I)\circ\delta_H$. For the normalisation condition, we recall that I is characterised by $I(T)\otimes 1 = V(1\otimes T)V^*$, and have a quick look at the definition of V to see that $\overline{V^*\xi} = V^*\bar{\xi}$. Then for $h, k\in\mathscr{H}$, $\xi,\eta\in L^2(G)$, and $\alpha,\beta\in L^2(H)$, an application of Lemma 1.9 gives

$$\begin{aligned}
(U(h\otimes\bar{\eta})\,|\,k\otimes\bar{\xi})(\alpha\,|\,\beta) &= (U_{12}(h\otimes\bar{\eta}\otimes\bar{\beta})\,|\,k\otimes\bar{\xi}\otimes\bar{\alpha})\\
&= (W_{13}(h\otimes V^*(\bar{\eta}\otimes\bar{\beta}))\,|\,k\otimes V^*(\bar{\xi}\otimes\bar{\alpha}))\\
&= (W_{13}(h\otimes\overline{V^*(\eta\otimes\beta)})\,|\,k\otimes\overline{V^*(\xi\otimes\alpha)})\\
&= (W_{13}^*(h\otimes V^*(\xi\otimes\alpha))\,|\,k\otimes V^*(\eta\otimes\beta))\\
&= (U^*(h\otimes\xi)\,|\,k\otimes\eta)(\alpha\,|\,\beta),
\end{aligned}$$

which proves that U satisfies (1.9), and ω is normalised.

THEOREM 3.3. *Let H be a closed subgroup of a locally compact group G, ω a normalised dual cocycle on H, and* $\operatorname{Inf}\omega = I \otimes I(\omega)$ *its inflation to a normalised dual cocycle on G. Let α^H denote the action of H by left translation on $A(H, \omega)$ and $C_r^*(\hat{H}, \omega)$, as in Proposition* 1.17. *For $f \in A(G, \operatorname{Inf}\omega)$, define $\Phi(f)\colon G \to A(H)$ by $\Phi(f)(x) = \alpha^G_{x^{-1}}(f)|_H$. Then Φ extends to an equivariant isomorphism of $(C_r^*(\hat{G}, \operatorname{Inf}\omega), G, \alpha^G)$ onto $(\operatorname{Ind}_H^G(C_r^*(\hat{H}, \omega), \alpha^H), G, \tau)$.*

Most of this section is devoted to the proof of this theorem, but before we start we look at the abelian case.

Remark 3.4. If we start with an abelian group G and a cocycle on G inflated from a cocycle σ on a quotient G/H, we can apply the theorem to the dual group $\hat{G}$. We first have to note that the cocycle σ is equivalent to a normalised one, and then let ω be the corresponding dual cocycle on $(G/H)^\wedge = H^\perp$. Since the Fourier transform carries the embedding $I\colon \mathscr{L}(H^\perp) \to \mathscr{L}(\hat{G})$ into the map $q^*\colon f \mapsto f \circ q$ of $L^\infty(G/H)$ into $L^\infty(G)$, the inflated cocycle $\operatorname{Inf}\sigma$ on G corresponds to the inflated dual cocycle $\operatorname{Inf}\omega$ on $\hat{G}$. Thus from Theorem 3.3 and Lemma 1.16 we deduce

$$\begin{aligned} C^*(G, \operatorname{Inf}\sigma) &\cong C_r^*(\hat{G}, \operatorname{Inf}\omega) \cong \operatorname{Ind}_{H^\perp}^{\hat{G}}(C_r^*(\widehat{H^\perp}, \omega), \alpha^{H^\perp}) \\ &\cong \operatorname{Ind}_{H^\perp}^{\hat{G}}(C^*(G/H, \sigma), \hat{\alpha}). \end{aligned} \tag{3.2}$$

This result about abelian groups contains a good deal of known information about the twisted group algebras of abelian groups. For example, since by [2, Theorem 3.1] every cocycle on G is equivalent to one inflated from a totally skew cocycle σ on a quotient G/H, and $\operatorname{Prim} C^*(G/H, \sigma)$ is then trivial [2], we deduce that $\operatorname{Prim} C^*(G, \operatorname{Inf}\sigma) \cong H^\perp$. This is a theorem of Green [8, Proposition 34] in general, and of Baggett–Kleppner in the type I case [2, p. 310]. Indeed, the isomorphism (3.2) may itself be known: the second author vaguely remembers being told something of the sorth by Dorte Olesen in 1986. We do not know where a proof of (3.2) has previously been written down, although a stable isomorphism could be easily obtained by converting $C^*(G/H, \sigma) \otimes \mathscr{K}$ into a crossed product $\mathscr{K} \times G/H$, as in [16, Section 3], and applying [15, Theorem 2.4]. It might be interesting to know if there is a similar stabilisation trick which will realise $C_r^*(\hat{G}, \omega) \otimes \mathscr{K}$ as a crossed product of $\mathscr{K}$ by a coaction of G; if so, the analogue of [15, Theorem 2.4] in [18] would presumably give a stable version of Theorem 3.3.

We begin the proof of Theorem 3.3 by noting from Lemma 3.1 that the restriction $\alpha^G_{x^{-1}}(f)|_H$ lies in $A(H)$, and hence in $C_r^*(\hat{H}, \omega)$. The map $\Phi(f)\colon C \to G_r^*(\hat{H}, \omega)$ is continuous because

$$\|\Phi(f)(x)\| \leqslant \|\alpha^G_{x^{-1}}(f)|_H\|_{A(H)} \leqslant \|\alpha^G_{x^{-1}}(f)\|_{A(G)} \tag{3.3}$$

(using Lemma 3.1), and translation is continuous on $A(G)$. If $h \in H$, then

$$\Phi(f)(xh) = \alpha^G_{h^{-1}x^{-1}}(f)|_H = \alpha^H_{h^{-1}}(\alpha^G_{x^{-1}}(f)|_H) = \alpha^H_{h^{-1}}(\Phi(f)(x)).$$

Since $A_c(G)$ is dense in $A(G)$, and $xH \mapsto \|\Phi(f)(x)\|$ has compact support when f does, the inequality (3.3) shows that $xH \mapsto \|\Phi(f)(x)\|$ lies in $C_0(G/H)$ for all $f \in A(G)$. Thus $\Phi(f)$ belongs to $\operatorname{Ind}^G_H(C^*_r(\hat{H}, \omega))$ for every $f \in A(G)$.

It is clear that Φ is $*$-linear, so we need to check multiplicativity. We compute:

$$\begin{aligned}
(\Phi(f)(x) *_\omega \Phi(g)(x))(h) &= \langle L^H_h, \alpha_x^{-1}(f)|_H *_\omega \alpha_x^{-1}(g)|_H\rangle \\
&= \langle (L^H_h \otimes L^H_h)\, \omega^*, \alpha_x^{-1}(f)\,|_H \otimes \alpha_x^{-1}(g)|_H\rangle \\
&= \langle I \otimes I((L^H_h \otimes L^H_h)\, \omega^*), \alpha_x^{-1}(f) \otimes \alpha_x^{-1}(g)\rangle \\
&= \langle (L^G_x \otimes L^G_x)(L^G_h \otimes L^G_h)(\operatorname{Inf} \omega)^*, f \otimes g\rangle \\
&= \langle L^G_x L^G_h, f *_{\operatorname{Inf} \omega} g\rangle \\
&= \langle I(L^H_h), \alpha_{x^{-1}}(f *_{\operatorname{Inf} \omega} g)\rangle \\
&= \langle L^H_h, \alpha_{x^{-1}}(f *_{\operatorname{Inf} \omega} g)|_H\rangle \\
&= (\Phi(f *_{\operatorname{Inf} \omega} g)(x))(h).
\end{aligned}$$

We shall complete the proof of Theorem 3.3 by showing that the homomorphism Φ has dense range and is isometric.

By Lemma 2.1, Φ will have dense range if the element $f \odot \phi$ has the form $\Phi(g)$ whenever $f \in A_c(G)$ and $\phi \in A_c(H)$. Given such f and ϕ, we define

$$g(x) = \int_H f(xh)\, \phi(h^{-1})\, dh;$$

since $h \mapsto f(\cdot h)\, \phi(h^{-1})$ is a continuous, compactly supported function with values in $A_c(G)$, g belongs to $A_c(G)$. For $k \in H$ we have

$$\alpha_x^{-1}(g)(k) = \int_H f(xkh)\, \phi(h^{-1})\, dh = \int_H f(xh)\, \phi(h^{-1}k)\, dh,$$

so that $\alpha_x^{-1}(g)|_H$ is given by the $A(H)$-valued integral $\int_H f(xh)\, \alpha^H_h(\phi)\, dh$ defining $(f \odot \phi)(x)$. Thus $\Phi(g) = f \odot \phi$, and the range of Φ is dense.

We shall prove that Φ is isometric by constructing a faithful representation π of $\operatorname{Ind} C^*_r(\hat{H}, \omega)$ such that $\pi \circ \Phi$ is equivalent to the regular representation μ^G of $C^*_r(\hat{G}, \operatorname{Inf} \omega)$. The representation π is the one induced from the regular representation μ^H of $C_r^*(\hat{H}, \omega)$: since (μ^H, R^H) is covariant (Proposition 1.17), Proposition 2.3 implies that $(\widetilde{\mu^H}, \operatorname{Ind}_H^G R^H)$ is a faithful covariant representation of $(\operatorname{Ind} C_r^*(\hat{H}, \omega), G, \tau)$. Because $Q^*(\operatorname{Ind} R^H) Q = R^G$, and we know (μ^G, R^G) is covariant, it is natural to guess that $Q^*(\widetilde{\mu^H} \circ \Phi) Q = \mu^G$, and this is precisely what happens. We need a technical lemma.

LEMMA 3.5. *For $T \in \mathscr{L}(H) \otimes \mathscr{L}(H)$ and $\xi, \eta, \phi, \psi \in C_c(G)$, we have*

$$\int_{G/H} \rho(x)^{-1} (i \otimes I(T)(Q\xi(x) \otimes \phi) \mid i \otimes I(W_H)(Q\eta(x) \otimes L_{x^{-1}}\psi))\, d\rho^\#(xH)$$
$$= (I \otimes I(T)(\xi \otimes \phi) \mid W_G(\eta \otimes \psi)). \qquad (3.4)$$

Proof. We first claim that it is enough to prove the equality for $T = L_a^H \otimes L_b^H$, where $a, b \in H$. The right-hand side is normal in T because $I \otimes I$ is, and we have to show that the same is true of the left-hand side. The integrand $f(T, x)$, say, is normal in T for each fixed x, and

$$|f(T, x)| \leqslant \rho(x)^{-1} \|T\| \, \|Q\xi(x) \otimes \phi\| \, \|Q\eta(x) \otimes \psi\|$$
$$\leqslant \rho(x)^{-1} \|T\| \, \|\phi\| \, \|\psi\| \, \|Q\xi(x)\| \, \|Q\eta(x)\|. \qquad (3.5)$$

Because $Q\xi$ and $Q\eta$ are in $\mathscr{H}_{\operatorname{Ind} R}$, both $\rho(x)^{-1/2} \|Q\xi(x)\|$ and $\rho(x)^{-1/2} \|Q\eta(x)\|$ are in $L^2(d\rho^\#)$, and the right-hand side of (3.5) is integrable. Now the Dominated Convergence Theorem implies that the left-hand side of (3.4) is normal in T.

So we suppose $T = L_a^H \otimes L_b^H$. Then the left-hand side of (3.4) is

$$\int_{G/H} \int_G \int_H \rho(x)^{-1} (L_a^H \otimes L_b^G)(Q\xi(x) \otimes \phi)(h, y)$$
$$\times \overline{(Q\eta(x) \otimes L_{x^{-1}}\psi)(h, h^{-1}y)}\, dh\, dy\, d\rho^\#(xH)$$
$$= \iiint \frac{\Delta(x^{-1})}{\rho(x)} \xi(a^{-1}hx^{-1})\, \phi(b^{-1}y)\, \overline{\eta(hx^{-1})\, \psi(xh^{-1}y)}\, dh\, dy\, d\rho^\#(xH)$$
$$= \int_{G/H} \int_H \frac{\Delta(x^{-1})}{\rho(x)} F(hx^{-1})\, dh\, d\rho^\#(xH),$$

where

$$F(z) = \xi(a^{-1}z)\, \overline{\eta(z)} \int_G \phi(b^{-1}y)\, \overline{\psi(z^{-1}y)}\, dy.$$

Thus the left-hand side of (3.4) is

$$\int_{G/H}\int_H \frac{\Delta(x^{-1})\,\delta(h^{-1})}{\rho(x)} F(h^{-1}x^{-1})\,dh\,d\rho^{\#}(xH)$$

$$= \int_{G/H}\int_H \rho(xh)^{-1}\,\Delta(h^{-1}x^{-1})\,F(h^{-1}x^{-1})\,dh\,d\rho^{\#}(xH)$$

$$= \int_G \rho(x)\,\Delta(x^{-1})\,F(x^{-1})\,\rho(x)^{-1}\,dx$$

$$= \int_G F(x)\,dx$$

$$= \int_G\int_G \xi(a^{-1}x)\,\overline{\eta(x)}\,\phi(b^{-1}y)\,\overline{\psi(x^{-1}y)}\,dx\,dy$$

$$= (L_a^G \otimes L_b^G(\xi\otimes\phi) \mid W_G(\eta\otimes\psi))$$

$$= (I\otimes I(L_a^H\otimes L_b^H)(\xi\otimes\phi) \mid W_G(\eta\otimes\psi)),$$

as required.

As foreshadowed above, we now prove that

$$Q^*(\widetilde{\mu^H}\circ\Phi)\,Q = \mu^G. \tag{3.6}$$

In the proof of Lemma 2.4 we showed that Q is unitary, so it is enough to check that, for $f\in A(G)$ and $\xi, \eta\in L^2(G)$, we have

$$(\widetilde{\mu^H}(\Phi(f))\,Q\xi \mid Q\eta) = (\mu^G(f)\,\xi \mid \eta). \tag{3.7}$$

The left-hand side of (3.7) is

$$\int_{G/H} \rho(x)^{-1}(\mu^H(\Phi(f)(x))\,Q\xi(x) \mid Q\eta(x))\,d\rho^{\#}(xH)$$

$$= \int_{G/H} \rho(x)^{-1}\,(\mu^H(\alpha^G_{x^{-1}}(f)|_H)\,Q\xi(x) \mid Q\eta(x))\,d\rho^{\#}(xH)$$

$$= \int_{G/H} \rho(x)^{-1}\langle W_H^*\Sigma(\omega^*), \omega_{Q\xi(x),\,Q\eta(x)}\otimes\alpha^G_{x^{-1}}(f)|_H\rangle\,\rho^{\#}(xH)$$

$$= \int_{G/H} \rho(x)^{-1}\,\langle i\otimes I(W_H^*\Sigma(\omega^*)), \omega_{Q\xi(x),\,Q\eta(x)}\otimes\alpha^G_{x^{-1}}(f)\rangle\,\rho^{\#}(xH),$$

because restriction: $A(G) \to A(H)$ is dual to I. If $f \in A(G)$ is the functional $\omega_{\phi,\psi}: T \mapsto (T\phi \mid \psi)$, then a calculation on elements of the form L_y shows that $\alpha_{x^{-1}}(f)$ is the functional $T \mapsto (T\phi \mid L_{x^{-1}}\psi)$, and the left-hand side of (3.7) is

$$\int_{G/H} \rho(x)^{-1}(i \otimes I(W_H^* \Sigma(\omega^*))(Q\xi(x) \otimes \phi) \mid Q\eta(x) \otimes L^G_{x^{-1}}\psi)\, d\rho^*(xH).$$

By Lemma 3.5, this is

$$\begin{aligned}(I \otimes I(\Sigma(\omega^*))(\xi \otimes \phi) \mid W_G(\eta \otimes \psi)) &= (W_G^* \Sigma(\operatorname{Inf}\omega^*)(\xi \otimes \phi) \mid (\eta \otimes \psi)) \\ &= \langle W_G^* \Sigma(\operatorname{Inf}\omega^*), \omega_{\xi,\eta} \otimes \omega_{\phi,\psi} \rangle \\ &= (S_f(W_G^* \Sigma(\operatorname{Inf}\omega^*))\, \xi \mid \eta) \\ &= (\mu^G(f)\, \xi \mid \eta),\end{aligned}$$

and we have proved (3.7), hence also (3.6).

From Eq. (3.6) we deduce that Φ is isometric for the norm on $C_r^*(\hat{G}, \operatorname{Inf}\omega)$, which is pulled back from the norm on $B(L^2(G))$ via μ^G, and the norm on $\operatorname{Ind} C_r^*(\hat{H}, \omega)$ induced from the norm on $C_r^*(\hat{H}, \omega)$. Since Proposition 1.17 says that (μ^G, R^G) is a covariant representation of $(C_r^*(\hat{G}, \operatorname{Inf}\omega), G, \alpha^G)$, Proposition 2.3 that $(\widetilde{\mu^H}, \operatorname{Ind} R^H)$ is a covariant representation of $(\operatorname{Ind} C_r^*(\hat{H}, \omega), G, \tau)$, and Lemma 2.4 that $Q^*(\operatorname{Ind} R^H)\, Q = R^G$, Eq. (3.6) also implies that the isomorphism Φ is equivariant. This completes the proof of Theorem 3.3.

EXAMPLE 3.6. (1) Let G be the real Heisenberg group, that is, $G = \mathbf{R}^3$ with multiplication given by

$$(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + y_1 x_2).$$

We fix a pair of real numbers μ, ν, and set

$$H_{\mu,\nu} = \{(\mu s, \nu s, t) : s, t \in \mathbf{R}\};$$

then $H_{\mu,\nu}$ is a closed abelian subgroup of G, isomorphic to $\mathbf{R}^2$ via $(\mu s, \nu s, t) \mapsto (s, t - \mu\nu s^2/2)$. (Indeed, $H_{\mu,\nu}$ is normal, and every proper normal, connected subgroup of G is of this form.) We can therefore take ω to be the dual cocycle on $H_{\mu,\nu}$ corresponding to the normalised cocycle σ_θ on $\mathbf{R}^2$ given by

$$\sigma_\theta((s_1, t_1), (s_2, t_2)) = \exp 2\pi i\theta(t_1 s_1 - s_1 t_2),$$

and our theorem says

$$C^*(\hat{G}, \operatorname{Inf}\omega) \cong \operatorname{Ind}^G_{H_{\mu,\nu}}(C^*(\hat{H}_{\mu,\nu}, \omega), \alpha^H) \cong \operatorname{Ind}^G_{H_{\mu,\nu}}(C^*(\mathbf{R}^2, \sigma_\theta), \hat{\alpha}),$$

where $\hat{\alpha}$ is the dual action of $H_{\mu,\nu} \cong \mathbf{R}^2$ on $C^*(\mathbf{R}^2, \sigma_\theta)$. Provided $\theta \neq 0$, $C^*(\mathbf{R}^2, \sigma_\theta) \cong \mathscr{K}(L^2(\mathbf{R}))$, and $\mathrm{Ind}_{H_{\mu,\nu}}^G C^*(\mathbf{R}^2, \sigma_\theta)$ is a separable continuous-trace C^*-algebra with spectrum $G/H_{\mu,\nu}$, and primitive quotients isomorphic to $\mathscr{K}(L^2(\mathbf{R}))$ [20]. Since the map $(x, y, z) \mapsto \nu x - \mu y$ induces an isomorphism of $G/H_{\mu,\nu}$ onto $\mathbf{R}$, and $\mathbf{R}$ is finite-dimensional with $H^3(\mathbf{R}, \mathbf{Z}) = 0$, it follows from [4; 17, 1.12] and the Dixmier–Douady Theorem [3, Chap. 10] that the algebra $\mathrm{Ind}_{H_{\mu,\nu}}^G C^*(\mathbf{R}^2, \sigma_\theta)$ is isomorphic to $C_0(\mathbf{R}, \mathscr{K}(L^2(\mathbf{R})))$.

(2) Let G be the quotient of the real Heisenberg group by the central copy of $\mathbf{Z}$, so that $G = \mathbf{R}^2 \times \mathbf{T}$ with multiplication given by

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 z_2 \exp 2\pi i y_1 x_2),$$

and let H be the quotient of $H_{\mu,\nu}$ by the same copy of $\mathbf{Z}$. So now $H \cong \mathbf{R} \times \mathbf{T}$, and we can take ω_θ to be the dual cocycle on H corresponding to the cocycle σ_θ on $\hat{H} = \mathbf{R} \times \mathbf{Z}$ defined by

$$\sigma_\theta((s_1, n_1), (s_2, n_2)) = \exp 2\pi i\theta(n_1 s_2 - s_1 n_2).$$

This dual cocycle is also inflated from the subgroup $\theta^{-1}\mathbf{Z} \times \mathbf{T}$, where the corresponding cocycle τ on $(\theta^{-1}\mathbf{Z} \times \mathbf{T})^\wedge \cong \mathbf{T} \times \mathbf{Z}$ has $C^*(\mathbf{T} \times \mathbf{Z}, \tau) \cong \mathscr{K}(L^2(\mathbf{T}))$; if we apply the analysis of the preceding example to this subgroup, we find that

$$C_r^*(\hat{G}, \mathrm{Inf}\,\omega_\theta) \cong \mathrm{Ind}_{\theta^{-1}\mathbf{Z} \times \mathbf{T}}^G \mathscr{K} \cong C_0(\mathbf{R} \times \mathbf{T}, \mathscr{K}).$$

However, if we want to view ω_θ as a one-parameter family of dual cocycles, it makes more sense to apply Theorem 3.3 to the fixed subgroup H, and then we get

$$C_r^*(\hat{G}, \mathrm{Inf}\,\omega_\theta) \cong \mathrm{Ind}_H^G C^*(\hat{H}, \sigma_\theta) \cong \mathrm{Ind}_H^G C^*(\mathbf{R} \times \mathbf{Z}, \sigma_\theta).$$

(3) Let $G = U_2(\mathbf{C})$, and take H to be the diagonal subgroup

$$H = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} : |\lambda| = |\mu| = 1 \right\} \cong \mathbf{T}^2.$$

The one-parameter family σ_θ of cocycles on $\mathbf{Z}^2 = \hat{\mathbf{T}}^2$ given by

$$\sigma_\theta((m_1, n_1), (m_2, n_2)) = \exp 2\pi i\theta(n_1 m_2 - m_1 n_2)$$

gives a one-parameter family ω_θ if dual cocycles on $U_2(\mathbf{C})$. Theorem 3.3 describes the dual-group algebra $C_r^*(\hat{U}_2, \mathrm{Inf}\,\omega_\theta)$ as $\mathrm{Ind}_H^{U_2} A_{2\theta}$, where $A_{2\theta} \cong C^*(\mathbf{Z}^2, \sigma_\theta)$ is the usual rotation algebra. (The factor of 2 comes in because we are using normalised cocycles; see Example 4.1 below.) So for irrational θ, this algebra has primitive ideal space U_2/H, but is definitely not type I.

4. Explicit Formulas for Dual Cocycles Inflated from Abelian Subgroups

The results of the previous sections show that we can construct a normalised dual cocycle on G from a normalised cocycle σ on the dual $\hat{H}$ of an abelian subgroup H, by first Fourier transforming σ to give a dual cocycle $\omega = \hat{\sigma}$ on $\hat{H}$, and then inflating ω to a dual cocycle $\operatorname{Inf} \omega$ on G. So far, this is the only known method of producing dual cocycles, and there is considerable evidence that they all arise this way. Indeed, the long-standing conjecture of [10] that a compact simple group G cannot act ergodically on an infinite-dimensional factor is by [23, 12] equivalent to showing that all dual cocycles are inflated from abelian subgroups. This is difficult stuff, though: Wassermann had to work very hard to establish this conjecture for $G = SU(2)$ [24] and other low-dimensional compact Lie groups [25]. In the Appendix, we show that a dual cocycle on a connected Lie group which lies on a one-parameter group of dual cocycles with a reasonable infinitesimal generator is necessarily inflated from an abelian subgroup.

So it makes sense to look in more detail at the structure of the algebras $A(G, \operatorname{Inf}_H^G \omega)$ and $C_r^*(\hat{G}, \operatorname{Inf}_H^G \omega)$, and since we know a great deal about 2-cocycles on abelian groups, we can obtain quite explicit formulas in the special case. Virtually every 2-cocycle on a locally compact abelian group $\hat{H}$ is equivalent to a continuous skew-symmetric bicharacter on $\hat{H} \times \hat{H}$, and all are if $s \mapsto 2s$ is an automorphism of $\hat{H}$ [11, p. 33] or if $\hat{H} = \mathbf{R}^p \times \mathbf{Z}^q$. The continuous bicharacters σ on $\hat{H} \times \hat{H}$ are in one-to-one correspondence with the continuous homomorphisms $\pi: s \mapsto \sigma(s, \cdot)$ of $\hat{H}$ into $H = H^{\wedge\wedge}$ [11, 1.1], and the skew-symmetry of σ is then equivalent to

$$\langle \pi(s), s \rangle = 1 \qquad \text{for all} \quad s \in \hat{H}. \tag{4.1}$$

We shall say that a continuous homomorphism $\pi: \hat{H} \to H$ is *skew* if it satisfies (4.1). Note that (4.1) implies

$$\langle \pi(s), t \rangle = \overline{\langle \pi(t), s \rangle} \qquad \text{for all} \quad s, t \in \hat{H};$$

to see this, expand the identity $\langle \pi(s+t), s+t \rangle = 1$.

Examples 4.1. (1) For $H = \mathbf{T}^2$ and $\theta \in \mathbf{R}$, we define $\pi_\theta: \hat{H} = \mathbf{Z} \times \mathbf{Z} \to H = \mathbf{T} \times \mathbf{T}$ by

$$\pi(m, n) = (\exp 2\pi i n\theta, \exp(-2\pi i m\theta)).$$

The corresponding cocycle on $\mathbf{Z}^2 = \hat{H}$ is then given by

$$\sigma_\theta((m, n), (p, q)) = \exp 2\pi i\theta(np - mq).$$

These are not all inequivalent: a quick calculation shows that if $\rho(m, n) = i^{m^2 - n^2}$, then

$$\sigma_{\theta+1/2}((m, n), (p, q)) = ((d\rho)\, \sigma_\theta)((m, n), (p, q))$$
$$:= \frac{\rho(m, n)\, \rho(p, q)}{\rho(m+p, n+q)} \sigma_\theta((m, n), (p, q)).$$

Thus the family $\{\sigma_\theta : \theta \in [0, \frac{1}{2})\}$ parametrises $H^2(\hat{H}, \mathbf{T})$, but the map $\theta \mapsto \sigma_\theta$ does not provide a canonical realisation of the 2-cocycles as skew-symmetric bicharacters. (The multiplier σ_θ is equivalent to the more common $\omega_{2\theta}((m, n), (p, q)) := \exp 2\pi i(2\theta)\, np$: take $\rho(m, n) = \exp 2\pi i\theta mn$, and then $\sigma_\theta = (d\rho)\, \omega_{2\theta}$. We do not use the representatives ω_θ for H^2 because they are not normalised.)

(2) Similar remarks apply to $H = \mathbf{T}^n$, $\hat{H} = \mathbf{Z}^n$: the skew-symmetric bicharacters are given by

$$\sigma_A((m, n), (p, q)) = \exp 2\pi i(n'Ap - m'Aq),$$

where A is a skew-symmetric matrix with real entries, and the class of σ_A in $H^2(\mathbf{Z}^n, \mathbf{T})$ is unchanged if we add multiples of $\frac{1}{2}$ to any entry of A [1].

(3) Let G be the quotient of the real Heisenberg group by the central copy of $\mathbf{Z}$, as in Example 3.6(2), and let H be the subgroup $H_{\mu,\nu} \cong \mathbf{R} \times \mathbf{T}$. Then, as in (1), the skew homomorphisms $\pi: \hat{H} = \mathbf{R} \times \mathbf{Z} \to H = \mathbf{R} \times \mathbf{T}$ are given by

$$\pi_\theta((r, n)) = (n\theta, \exp(-2\pi i\theta r))$$

for some $\theta \in \mathbf{R}$.

Suppose that G is a locally compact group, H is a closed abelian subgroup, $\pi: \hat{H} \to H$ is a continuous skew homomorphism, and σ is the cocycle on $\hat{H}$ defined by

$$\sigma(s, t) = \langle \pi(s), t \rangle \qquad \text{for} \quad s, t \in \hat{H}.$$

The corresponding dual cocycle ω on H is given by

$$\langle \omega, \phi \otimes \psi \rangle = \int_{\hat{H}} \int_{\hat{H}} \sigma(s, t)\, \hat{\phi}(s)\, \hat{\psi}(t)\, ds\, dt \qquad \text{for} \quad \phi, \psi \in A(H),$$

where $\hat{\phi}$ is by definition the unique element of $L^1(\hat{H})$ whose inverse Fourier transform is ϕ; if $\phi \in A(H) \cap L^1(H)$ (if $\phi \in A_c(H)$, for example), then $\hat{\phi}$ is given by the usual formula

$$\hat{\phi}(s) = \int_H \phi(h)\, \overline{\langle h, s \rangle}\, dh.$$

The multiplication on $A(G, \operatorname{Inf}\omega)$ is given by

$$\begin{aligned} f *_{\operatorname{Inf}\omega} g(x) &= \langle L_x^G, f *_{\operatorname{Inf}\omega} g\rangle \\ &= \langle (L_x^G \otimes L_x^G)(\operatorname{Inf}\omega)^*, f \otimes g\rangle \\ &= \langle I \otimes I(\omega^*), \alpha_{x^{-1}}(f) \otimes \alpha_{x^{-1}}(g)\rangle \\ &= \langle \omega^*, \alpha_{x^{-1}}(f)|_H \otimes \alpha_{x^{-1}}(g)|_H\rangle \\ &= \int_{\hat{H}} \int_{\hat{H}} \overline{\sigma(s, t)}\, \hat{f}(x, s)\, \hat{g}(x, t)\, ds\, dt, \end{aligned}$$

where now $\hat{f}(x, \cdot)$ is the unique element of $L^1(\hat{H})$ whose inverse Fourier transform is the function $h \mapsto f(xh)$; $\hat{f}$ exists because the restriction $\alpha_{x^{-1}}(f)|_H$ is in $A(H)$, and if it is also in $L^1(H)$, then $\hat{f}$ is the partial Fourier transform

$$\hat{f}(x, s) = \int_H f(xh)\, \overline{\langle h, s\rangle}\, dh.$$

Thus we have

$$f *_{\operatorname{Inf}\omega} g(x) = \int_{\hat{H}} \int_{\hat{H}} \overline{\langle \pi(s), t\rangle}\, \hat{f}(x, s)\, \hat{g}(x, t)\, ds\, dt. \tag{4.2}$$

Since $\overline{\langle \pi(s), t\rangle} = \langle s, \pi(t)\rangle$, the inside integral is an inverse Fourier transform, and we can rewrite (4.2) as

$$f *_{\operatorname{Inf}\omega} g(x) = \int_{\hat{H}} f(x\pi(t))\, \hat{g}(x, t)\, dt.$$

Alternatively, we can use Fubini's theorem to change the order of integration in (4.2), and obtain

$$f *_{\operatorname{Inf}\omega} g(x) = \int_{\hat{H}} \hat{f}(x, s)\, g(x\pi(-s))\, ds.$$

When the subgroup H is a direct product $H_1 \times H_2$, as in our examples, the skew homomorphism $\pi\colon \hat{H} \to H$ has the form

$$\pi(h_1, h_2) = (\pi_{11}(h_1)\, \pi_{12}(h_2), \pi_{21}(h_1)\, \pi_{22}(h_2)),$$

where $\pi_{ij}\colon \hat{H}_j \to H_i$, π_{11} and π_{22} are skew, and

$$\langle \pi_{21}(s), t\rangle = \overline{\langle s, \pi_{12}(t)\rangle} = \langle s, \pi_{12}(-t)\rangle \qquad \text{for} \quad s \in \hat{H}_1, \quad t \in \hat{H}_2; \tag{4.3}$$

we write $\pi_{21}(s)=\pi'_{12}(-s)$ as shorthand for (4.3). In our examples, the diagonal terms π_{ii} are absent and π is determined by a continuous homomorphism $\rho: \hat{H}_2 \to H_1$ via

$$\pi(s_1, s_2)=(\rho(s_2), \rho'(-s_1)).$$

For such π the formula (4.2) can again be substantially simplified,

$$\begin{aligned} f *_{\mathrm{Inf}\,\omega} g(x) &= \int_{\hat{H}_2}\int_{\hat{H}_2}\int_{\hat{H}_1}\int_{\hat{H}_1} \langle \rho'(s_1), t_2\rangle \overline{\langle \rho(s_2), t_1\rangle} \\ &\qquad \times \hat{f}(x, s_1, s_2)\, \hat{g}(x, t_1, t_2)\, ds_1\, dt_1\, ds_2\, dt_2 \\ &= \int_{\hat{H}_2}\int_{\hat{H}_2} \hat{f}(x(\rho(t_2), 0), s_2)\, \hat{g}(x(\rho(-s_2), 0), t_2)\, ds_2\, dt_2, \end{aligned}$$

where now $\hat{f}(y, \cdot)$ is the unique function in $L^1(H_2)$ with inverse Fourier transform $h_2 \mapsto f(y(0, h_2))$. Note that, by Fubini's Theorem, the integrand in the last integral is integrable because the integrand in the previous line is integrable in (s_1, s_2, t_1, t_2).

For our final simplification, we supress all mention of H_1 by starting with an abelian subgroup K of G and a continuous homomorphism $\rho: \hat{K} \to G$ such that $\rho(s)\, k = k\rho(s)$ for all $s \in \hat{K}$, $k \in K$, and $\overline{\rho(\hat{K})} \cap K = \varnothing$; we can then take $H_2 = K$, $H_1 = \overline{\rho(\hat{K})}$ and, for example, replace $(\rho(t_2), 0)$ by $\rho(t_2)$. We sum up our calculations:

First, a convention. If K is a closed abelian subgroup of G and $f \in A(G)$, then we denote by $\hat{f}(x, \cdot)$ the unique element of $L^1(\hat{K})$ whose inverse Fourier transform is the function $\alpha_{x^{-1}}(f)|_K: k \mapsto f(xk)$. Our earlier comments about (3.3) show that the map $x \mapsto \alpha_{x^{-1}}(f)|_K$ is continuous from G to $A(K) \cong L^1(\hat{K})$. We therefore have:

LEMMA 4.2. *If $f \in A(G)$ there is a function $\hat{f}: G \times \hat{K} \to \mathbf{C}$ such that $x \mapsto \hat{f}(x, \cdot)$ is in $C_b(G, L^1(\hat{K}))$, $k \mapsto f(xk)$ is the inverse Fourier transform of $s \mapsto \hat{f}(x, s)$, and $\sup_x \|\hat{f}(x, \cdot)\|_{L^1(\hat{K})} \leqslant \|f\|_{A(G)}$. (If $\alpha_{x^{-1}}(f)|_K \in A(K) \cap L^1(K)$, then*

$$\hat{f}(x, s) = \int_K f(xk)\, \overline{\langle k, s\rangle}\, dk.)$$

In any case, we refer to $\hat{f}(x, \cdot)$ as the partial Fourier transform of f along xK.

THEOREM 4.3. *Suppose G is a locally compact group, K a closed abelian subgroup, and $\rho: \hat{K} \to G$ a continuous homomorphism such that $\rho(s)\, k = k\rho(s)$*

for all $s \in \hat{K}$, $k \in K$, *and* $\overline{\rho(\hat{K})} \cap K = \varnothing$. *Then* $H = \overline{\rho(\hat{K})} \times K$ *is a closed abelian subgroup of* G; *let* ω *be the normalised dual cocycle on* H *corresponding to the normalised cocycle* σ *on* $\hat{H}$ *given by*

$$\sigma((s_1, s_2), (t_1, t_2)) = \langle \rho(s_2), t_1 \rangle \overline{\langle s_1, \rho(t_2) \rangle} \qquad \textit{for} \quad (s_1, s_2), (t_1, t_2) \in \hat{H}.$$

Then the multiplication on $A(G, \operatorname{Inf}\omega)$ *is given by*

$$f *_{\operatorname{Inf}\omega} g(x) = \int_{\hat{K}} \int_{\hat{K}} \hat{f}(x\rho(t), s)\, \hat{g}(x\rho(-s), t)\, ds\, dt,$$

where $\hat{f}(y, \cdot)$ *denotes the partial Fourier transform of* f *along* yK, *as above.*

Example 4.4. Return to the Heisenberg group example of 3.6(1), with K the central copy of $\mathbf{R}$, and $\rho: \mathbf{R} \to G$ given by $\rho(t) = (\mu t, \nu t, \mu\nu t^2/2)$. The identity $\hat{f}(xl, s) = \langle l, s \rangle \hat{f}(x, s)$ means we can view the partial Fourier transform as a function on $\mathbf{R}^2 \times \mathbf{R} = \mathbf{R}^2 \times \hat{\mathbf{R}}$ via

$$\hat{f}(x, y, t) = \int_{\mathbf{R}} f(x, y, z)\, e^{-2\pi i z t}\, dz,$$

and the same identity allows us to express the multiplication on $A(G)$ as

$$f * g(x, y, z) = \int_{\mathbf{R}} \int_{\mathbf{R}} \exp 2\pi i \left(z(s+t) + s\left(\frac{\mu\nu t^2}{2} + y\mu t\right) + t\left(\frac{\mu\nu s^2}{2} - y\mu s\right)\right)$$
$$\times \hat{f}(x + \mu t, y + \nu t, s)\, \hat{g}(x - \mu s, y - \nu s, t)\, ds\, dt.$$

Remark. We have seen that G always acts by left translation on $A(G, \omega)$, but since the definition of $*_\omega$ involves putting ω^* on the right of the variable, we would not necessarily expect G to act by right translation on $A(G, \omega)$. However, we can identity some situations in which it does, and these will be important later. We begin by looking at the effect of automorphisms and obtain information about right translation by x on G by observing that $y \mapsto yx = x(x^{-1}yx)$ is the composition of an inner automorphism and a left translation.

If α is a bicontinuous automorphism of G (write $\alpha \in \operatorname{Aut} G$), there is a constant $\Delta(\alpha)$ such that $\int_G f \circ \alpha = \Delta(\alpha) \int_G f$, and then $U_\alpha \xi = \Delta(\alpha)^{1/2}\, \xi \circ \alpha$ defines a unitary operator U_α on $L^2(G)$. Since we trivially have $L_x U_\alpha = U_\alpha L_{\alpha(x)}$, the map $T \mapsto U_\alpha^* T U_\alpha$ is a normal isomorphism $\bar{\alpha}$ of $\mathscr{L}(G)$ onto $\mathscr{L}(G)$ such that $\bar{\alpha}(L_x) = L_{\alpha(x)}$.

Lemma 4.5. *Suppose* ω *is a normalised dual cocycle and* $\alpha \in \operatorname{Aut} G$. *Then* $\alpha^*: f \mapsto f \circ \alpha$ *is an automorphism of* $A(G, \omega)$ *if and only if* $\bar{\alpha} \otimes \bar{\alpha}(\omega) = \omega$. *In the case,* α^* *extends to a* $*$*-automorphism of* $C_r^*(\hat{G}, \omega)$, *implemented by* U_α^*.

Proof. Since α^* is always a bijection, we have only to show that it is multiplicative iff $\bar{\alpha}\otimes\bar{\alpha}(\omega)=\omega$. But it is easy to check that $\langle \bar{\alpha}(T), f\rangle = \langle T, f\circ\alpha\rangle$, so

$$\begin{aligned}
&(f\circ\alpha) *_\omega (g\circ\alpha)(x) = (f *_\omega g)\circ\alpha(x)\\
&\quad\Leftrightarrow \langle (L_x\otimes L_x)\,\omega^*, (f\circ\alpha)\otimes(g\circ\alpha)\rangle = \langle (L_{\alpha(x)}\otimes L_{\alpha(x)})\,\omega^*, f\otimes g\rangle\\
&\quad\Leftrightarrow \langle \bar{\alpha}\otimes\bar{\alpha}(L_x\otimes L_x)\,\bar{\alpha}\otimes\bar{\alpha}(\omega^*), f\otimes g\rangle = \langle (L_{\alpha(x)}\otimes L_{\alpha(x)})\,\omega^*, f\otimes g\rangle\\
&\quad\Leftrightarrow \langle (L_{\alpha(x)}\otimes L_{\alpha(x)})\,\bar{\alpha}\otimes\bar{\alpha}(\omega^*), f\otimes g\rangle = \langle (L_{\alpha(x)}\otimes L_{\alpha(x)})\,\omega^*, f\otimes g\rangle.
\end{aligned}$$

This is true for all $x\in G$ and $f, g\in A(G)$ precisely when $\bar{\alpha}\otimes\bar{\alpha}(\omega)=\omega$.

For the last part, we let μ denote the regular representation of $A(G,\omega)$; then we need to show that (μ, U_α) us a covariant representation of $(A(G,\omega),\alpha^*)$. Observe that $\langle f\circ\alpha, T\rangle = \langle f, \bar{\alpha}(T)\rangle = \langle f, U_\alpha^* T U_\alpha\rangle$, and that $U_\alpha\otimes U_\alpha$ commutes with W_G^*. Then, writing U for U_α, we have

$$\begin{aligned}
U^*\mu(f\circ\alpha)\,U &= S_{f\circ\alpha}((U^*\otimes 1)\,W_G^*\Sigma(\omega^*)(U\otimes 1))\\
&= S_f((U^*\otimes U^*)\,W_G^*\Sigma(\omega^*)(U\otimes U))\\
&= S_f(W_G^*(U^*\otimes U^*)\,\Sigma(\omega^*)(U\otimes U))\\
&= S_f(W_G^*\Sigma(\bar{\alpha}\otimes\bar{\alpha}(\omega^*)))\\
&= S_f(W_G^*\Sigma(\omega^*))\\
&= \mu(f),
\end{aligned}$$

as required.

LEMMA 4.6. *Suppose G is a second countable locally compact group, H a closed abelian subgroup, and ω is the normalised dual cocycle on H given by a skew homomorphism $\pi\colon \hat{H}\to H$. Suppose $\alpha\in\operatorname{Aut} G$ satisfies $\alpha(H)=H$ and $\alpha\circ\pi(s\circ\alpha)=\pi(s)$ for all $s\in\hat{H}$. Then $f\mapsto f\circ\alpha$ is an automorphism of $A(G,\operatorname{Inf}\omega)$ and extends to a *-automorphism of $C_r^*(\hat{G},\operatorname{Inf}\omega)$.*

Proof. Since $\bar{\alpha}$, $\overline{\alpha_H} := \overline{\alpha|_H}$, and I are all normal, the calculation

$$\bar{\alpha}\circ I(L_h^H) = \bar{\alpha}(L_h^G) = L_{\alpha(h)}^G = I(L_{\alpha(h)}^H) = I\circ\overline{\alpha_H}(L_h^H)$$

implies that

$$\begin{aligned}
\bar{\alpha}\otimes\bar{\alpha}(\operatorname{Inf}\omega) &= \bar{\alpha}\otimes\bar{\alpha}(I\otimes I(\omega)) = I\otimes I(\overline{\alpha_H}\otimes\overline{\alpha_H}(\omega))\\
&= (I\otimes I)\circ(\overline{\alpha_H}\otimes\overline{\alpha_H})\circ\operatorname{Ad}(\mathscr{F}\otimes\mathscr{F})(\sigma),
\end{aligned}$$

where $\mathscr{F}: L^2(\hat{H}) \to L^2(H)$ is the Fourier transform as before and $\sigma(s, t) = \langle \pi(s), t \rangle$. Now $\mathscr{F}^{-1} L_h F$ is the function $f_h: t \mapsto \overline{\langle h, t \rangle}$, and hence

$$\overline{\alpha_H}(\mathscr{F} f_h \mathscr{F}^{-1}) = \overline{\alpha_H}(L_h^H) = L_{\alpha(h)}^H = \mathscr{F} f_{\alpha(h)} \mathscr{F}^{-1} = \mathscr{F}(f_h \circ \alpha') \mathscr{F}^{-1},$$

where $\langle \alpha'(t), h \rangle := \langle t, \alpha(h) \rangle$. Again, we can use the normality of $\bar{\alpha}$ and $(\alpha')^*$ to deduce that $\overline{\alpha_H} \circ \mathrm{Ad}\, \mathscr{F} = \mathrm{Ad}\, \mathscr{F} \circ (\alpha')$, and

$$\bar{\alpha} \otimes \bar{\alpha}(\mathrm{Inf}\, \omega) = (I \otimes I)\, \mathrm{Ad}(\mathscr{F} \otimes \mathscr{F})(\sigma \circ (\alpha' \otimes \alpha')).$$

But a simple calculation shows that $\alpha \circ \pi \circ \alpha' = \pi \Leftrightarrow \sigma \circ (\alpha' \otimes \alpha') = \sigma$, so

$$\bar{\alpha} \otimes \bar{\alpha}(\mathrm{Inf}\, \omega) = (I \otimes I)\, \mathrm{Ad}(\mathscr{F} \otimes \mathscr{F})(\sigma) = \mathrm{Inf}\, \omega,$$

and the result follows from Lemma 4.5.

COROLLARY 4.7. *If $x \in G$ satisfies $x^{-1} H x = H$ and $\pi(s \circ \mathrm{Ad}\, x^{-1}) = \mathrm{Ad}\, x \circ \pi(s)$, then $\beta_x(f)(y) = f(yx)$ is an automorphism of $A(G, \mathrm{Inf}\, \omega)$.*

Proof. In these circumstances, β_x is the composition of $f \mapsto f \circ \mathrm{Ad}\, x^{-1}$ and the left translation α_x.

EXAMPLE 4.8. We return to Example 3.6(2), where G is the modified Heisenberg group, $H = H_{\mu, \nu}$, and $\pi: \mathbf{R} \times \mathbf{Z} = \hat{H} \to H$ is defined via the isomorphism of H with $\mathbf{R} \times \mathbf{T}$ by $\pi(r, n) = (n\theta, \exp(-2\pi i r \theta))$. We identify $(s, z) \in \mathbf{R} \times \mathbf{T}$ with $(\mu s, \nu s, z \exp(-\pi i \mu \nu s^2))$, and compute

$$\begin{aligned}(x, y, 1)^{-1}(s, z)(x, y, 1) &= (\mu s, \nu s, z \exp(-\pi i \mu \nu s^2 - 2\pi i s(\mu y - \nu x)))\\ &= (s, z \exp(-2\pi i s(\mu y - \nu x))).\end{aligned}$$

Thus

$$\begin{aligned}\langle (s, z), (r, n) \circ \mathrm{Ad}(x, y, 1)^{-1} \rangle &= \exp(2\pi i r s)\, z^n \exp(-2\pi i s(\mu y - \nu x)\, n)\\ &= \langle (s, z), (r - (\mu y - \nu x)\, n, n) \rangle,\end{aligned}$$

so that

$$\pi((r, n) \circ \mathrm{Ad}(x, y, 1)^{-1}) = (n\theta, \exp - 2\pi i(r - n(\mu y - \nu x))\, \theta).$$

Now one easily checks that this is $\mathrm{Ad}(x, y, 1)^{-1}(\pi(r, n))$. Thus according to Corollary 4.7, right translation gives an action of G on $A(G, \mathrm{Inf}\, \omega)$, which extends to an action β of G on $C_r^*(\hat{G}, \mathrm{Inf}\, \omega)$; it is continuous because we can write down a unitary representation on $L^2(G)$ implementing it.

5. Dense Subalgebras of $C^*(\hat{G}, \operatorname{Inf} \omega)$

When forming deformations of $C_0(X)$, one wants a *-subspace $\mathscr{A}$ of $C_0(X)$ and a one-parameter family of multiplications on $\mathscr{A}$. We have seen that, when $X = G$, $A(G)$ is a suitable subspace. To deform algebras of the form $C_0(G/\Gamma)$, it is tempting to look for a suitable Γ-invariant subalgebra of $C_0(G)$, but in general there will be no Γ-invariant functions which vanish at infinity, and one is forced to look in $C_b(G)$ rather than $C_0(G)$. The obvious algebra of bounded functions associated to the Fourier algebra $A(G)$ is $B(G)$, but unfortunately $B(G) \cap C_0(G/\Gamma)$ need not be large enough to separate points of G/Γ (see Example 5.4 below). Here we propose another *-space of functions in $C_0(G)$, which is a Banach *-algebra with respect to the same multiplication as $A(G, \operatorname{Inf} \omega)$, but whose bounded analogue will contain more Γ-invariant functions. The basic observation is that, when the abelian subgroup K in Theorem 4.3 is compact, the partial Fourier coefficients make sense for any continuous function, and asking that $f|_K$ be in $A(K)$ is equivalent to asserting that the partial Fourier series of f be summable.

Lemma 5.1. *Suppose G is a locally compact group and K a compact abelian subgroup. For any $f \in C_b(G)$, we define the partial Fourier transform $\hat{f}\colon G \times \hat{K} \to \mathbf{C}$ by*

$$\hat{f}(x, s) = \int_K \overline{\langle k, s\rangle}\, f(xk)\, dk.$$

Then

(1) $\hat{f}(\cdot, s) \in C_b(G)$ *for all* $s \in \hat{K}$;

(2) $\hat{f}(xk, s) = \langle k, s\rangle \hat{f}(x, s)$ *for* $k \in K$, $s \in \hat{K}$;

(3) *if* $f(xk) = \langle k, s\rangle f(x)$ *for* $k \in K$ *and* $s \in \hat{K}$, *then* $\hat{f}(\cdot, t) = 0$ *for* $t \neq s$, *and* $\hat{f}(\cdot, s) = f$;

(4) *if* $s \in G$ *is fixed and* $k \mapsto f(xk)$ *is in* $A(K)$ (*in particular, for all* x *if* $f \in A(G)$), *then* $f(x) = \sum_{s \in \hat{K}} \hat{f}(x, s)$.

Proof. (1) follows from a standard compactness argument, (2) and (3) from routine computations, and (4) from the Fourier Inversion theorem.

Proposition 5.2. *Let G be a locally compact group, K a compact abelian subgroup, and $p\colon \hat{K} \to G$ a continuous homomorphism such that $p(s)k = kp(s)$ for $k \in K$, $s \in \hat{K}$. Then*

$$C_{b,1}(G) := \left\{ f \in C_b(G) \colon \|f\|_{\infty,1} := \sum_s \sup_x |\hat{f}(x, s)| < \infty \right\}$$

*is a Banach *-algebra with norm* $\|\cdot\|_{\infty,1}$ *and operations*

$$\begin{aligned} f * g(x) &= \sum_{s,t} \hat{f}(x\rho(t), s)\, \hat{g}(x\rho(-s), t), \\ f^*(x) &= \overline{f(x)}. \end{aligned} \tag{5.1}$$

Further, $C_{0,1}(G) := C_{b,1}(G) \cap C_0(G)$ *is a closed 2-sided ideal in* $C_{b,1}(G)$.

Proof. If $f \in C_{b,1}(G)$, we can view $\hat{f}$ as an element $s \mapsto \hat{f}(\cdot, s)$ of $l^1(\hat{K}, C_b(G))$, and $f \mapsto \hat{f}$ is then an isometric embedding of $C_{b,1}(G)$ in the Banach space $l^1(\hat{K}, C_b(G))$. Thus if $\{f_n\}$ is Cauchy in $C_{b,1}(G)$, the sequence $\{\hat{f}_n\}$ converges in $l^1(\hat{K}, C_b(G))$, say $\hat{f}_n \to g$. We define $f(x) = \sum_s g(x, s)$, which converges absolutely and uniformly to a continuous function f with $\|f\|_\infty \leqslant \|g\|_{\infty,1}$. It follows easily from Lemma 5.1(2) that $g(xk, s) = \langle k, s\rangle\, g(x, s)$, and since $\sum_s g(x, s)$ converges uniformly in x, we deduce from Lemma 5.1(3) that

$$\hat{f}(x, s) = \sum_t (g(\cdot, t))^\wedge\, (x, s) = \hat{g}(x, s).$$

Thus $f \in C_{b,1}(G)$, and $\hat{f}_n \to g$ in $l^1(\hat{K}, C_b(G))$ is equivalent to $f_n \to f$ in $C_{b,1}(G)$.

Next, observe that

$$\begin{aligned} \sum_{s,t} |\hat{f}(x\rho(t), s)\, \hat{g}(x\rho(-s), t)| &\leqslant \sum_s \left(\|\hat{f}(\cdot, s)\|_\infty \sum_t |\hat{g}(x\rho(-s), t)| \right) \\ &\leqslant \sum_s \|\hat{f}(\cdot, s)\|_\infty \sum_t \|\hat{g}(\cdot, t)\|_\infty \\ &= \|f\|_{\infty,1}\, \|g\|_{\infty,1}, \end{aligned}$$

so the series defining $f * g$ converges absolutely, uniformly in x, to a continuous function $f * g$ on G. We can then pull the summation through the integral defining $(f * g)^\wedge\,(x, r)$ to see that

$$(f * g)^\wedge\,(x, r) = \sum_p \hat{f}(x\rho(r-p), p)\, \hat{g}(x\rho(-p), r-p).$$

Thus we have

$$\begin{aligned} \|f * g\|_{\infty,1} &= \sum_r \sup_x \left| \sum_p \hat{f}(x\rho(r-p), p)\, \hat{g}(x\rho(-p), r-p) \right| \\ &\leqslant \sum_r \sum_p \|\hat{f}(\cdot, p)\|_\infty\, \|\hat{g}(\cdot, r-p)\|_\infty \\ &= \|f\|_{\infty,1}\, \|g\|_{\infty,1}. \end{aligned}$$

A messy and boring calculation verifies that $(f * g) * h = f * (g * h)$, and we have shown that $C_{b,1}(G)$ is a Banach algebra. That $f \mapsto f^*$ is an involution follows from the easily checked identity $\widehat{f^*}(x, s) = \overline{\hat{f}(x, -s)}$.

We have already seen that the series defining $f * g$ converges uniformly in x; if f or g is in $C_0(G)$, then each summand is also in $C_0(G)$, and hence so is the sum $f * g$. Thus $C_{0,1}(G)$ is an ideal. To see that it is closed, note first that for every $f \in C_{b,1}(G)$ and $x \in G$, we can recover the function $k \mapsto f(xk)$ as the sum of the absolutely convergent Fourier series $\sum_s \hat{f}(x, s)\langle k, s\rangle$, and we have $\|f\|_\infty \leqslant \|f\|_{\infty,1}$. Thus if $f_n \to g$ in $C_{b,1}(G)$, and each f_n is in $C_0(G)$, we have $f_n \to g$ uniformly, and g also lies in $C_0(G)$.

Remark. The subspace $C_{c,1}(G) = C_{b,1}(G) \cap C_c(G)$ is *not* an ideal in $C_{b,1}(G)$. To see this, suppose f has compact support C. Then $x \mapsto \hat{f}(x\rho(t), s)$ has support in $C\rho(t)^{-1}$, so the summands in (5.1) have different supports. If $\rho(\hat{K})$ is not compact in G, which is easy to arrange if K is the subgroup $\{0\} \times \mathbf{T}$ of $\mathbf{R} \times \mathbf{T}$, for example, then $\bigcup_{t \in \hat{K}} C\rho(t)^{-1}$ will not be compact in G. Thus to force $f * g$ into $C_c(G)$, we would need to insist that $\hat{f}$ and $\hat{g}$ have finite support in the $\hat{K}$-variable.

PROPOSITION 5.3. *Suppose G, K, and $\rho\colon \hat{K} \to G$ are as in Proposition 5.2. Then $A_1(G) = A(G) \cap C_{b,1}(G)$ is a dense *-subalgebra of both $A(G, \operatorname{Inf} \omega)$ and $C_{0,1}(G, \operatorname{Inf} \omega)$. If K is central in G, then $A(G)$ is itself a dense *-subalgebra of $C_{0,1}(G, \operatorname{Inf} \omega)$.*

Proof. Since the multiplication formulas are the same, we have to show only that $A_1(G)$ is dense in $A(G)$ and $C_{0,1}(G)$. For the first, we start with $\xi, \eta \in C_c(G)$, $f(x) = (L_x^G \xi \mid \eta)$, and we choose $\phi \in L^1(K)$ such that $\hat{\phi} \in l^1(\hat{K})$ and $L^G|_K(\phi)\,\xi$ approximates ξ in $L^2(G)$. If we now set $g(x) = (L_x^G L^G|_K(\phi)\,\xi \mid \eta)$, then g approximates f in $A(G)$. But a quick calculation shows

$$g(x) = \int_K \phi(h)\, f(xh)\, dh,$$

and hence

$$\begin{aligned}\hat{g}(x, s) &= \int_K \overline{\langle k, s\rangle} \int_K \phi(h)\, f(xkh)\, dh\, dk \\ &= \int_K \int_K \overline{\langle kh, s\rangle}\, \overline{\langle h, -s\rangle}\, \phi(h)\, f(xkh)\, dh\, dk \\ &= \hat{\phi}(-s)\, \hat{f}(x, s).\end{aligned}$$

Thus for any $x \in G$, we have

$$|\hat{g}(x, s)| \leqslant |\hat{\phi}(-s)| \int_K |f(xk)|\, dk \leqslant |\hat{\phi}(-s)|\, \|f\|_\infty,$$

so that $\|g\|_{\infty,1} \leqslant \|\hat{\phi}\|_1\, \|f\|_\infty < \infty$. Thus $g \in A_1(G)$, and we have shown $A_1(G)$ dense in $A(G)$.

To see that $A_1(G)$ is dense in $C_{0,1}(G)$, we start with $f \in C_{0,1}(G)$. Since we can approximate f by a finite sum $\sum_s \hat{f}(\cdot, s)$, it will be enough to approximate one fixed $\hat{f}(\cdot, s)$ by something in $A_1(G)$. We can certainly find $g \in A(G)$ such that $\|g - \hat{f}(\cdot, s)\|_\infty < \varepsilon$. Then $g_1(x) = \hat{g}(x, s)$ is in $A(G)$ because it is the integral of the continuous $A(G)$-valued function $k \mapsto \overline{\langle k, s\rangle}\, g(\cdot k)$, and a trivial estimate shows $\|g_1 - \hat{f}(\cdot, s)\|_\infty < \varepsilon$. Since $\hat{g}_1(\cdot, t) = 0$ unless $t = s$, Lemma 5.1(3) implies that $\|g_1 - \hat{f}(\cdot, s)\|_{\infty,1} = \|g_1 - \hat{f}(\cdot, s)\|_\infty$, and we are done.

We now show that $A(G) \subset C_{0,1}(G)$ if K is central. First note that if $f(x) = (L_x \xi \mid \eta)$, then

$$\hat{f}(x, s) = \int_K \overline{\langle k, s\rangle} (L_k^G \xi \mid L_{x^{-1}}^G \eta)\, dk = (L_x^G P_s \xi \mid \eta), \tag{5.2}$$

where P_s is the family of mutually orthogonal projections on $L^2(G)$ defined by

$$(P_s \xi \mid \eta) = \int_K \overline{\langle k, s\rangle} (L_k^G \xi \mid \eta)\, dk.$$

Because K is central and the operators P_s are projections, we have $L_x^G P_s = L_x^G P_s^2 = P_s L_x^G P_s$, and

$$\hat{f}(x, s) = (L_x^G P_s \xi \mid \eta) = (P_s L_x^G P_s \xi \mid \eta) = (L_x^G P_s \xi \mid P_s \eta).$$

It follows that

$$\sum_s \sup_x |\hat{f}(x, s)| \leqslant \sum_s \|P_s \xi\|\, \|P_s \eta\| \leqslant \left(\sum_s \|P_s \xi\|^2\right)^{1/2} \left(\sum_s \|P_s \eta\|^2\right)^{1/2} \leqslant \|\xi\|\, \|\eta\|.$$

Thus $\|f\|_{\infty,1} \leqslant \|\xi\|\, \|\eta\|$, and $f \in C_{0,1}(G)$. This completes the proof of the proposition.

Concluding remarks. This construction suggests that we need not have started with a group G: a space X with a right action of (say) $\mathbf{T}^2$ would have been enough. However, even this extra generality does not cover some of the potentially interesting examples. For example, if G is the real Heisenberg group and Γ its integer subgroup, [21] suggests we should be able to deform $C(G/\Gamma)$ by taking K to be the copy of $\mathbf{T}$ coming from the centre of G, and $\rho: \mathbf{Z} = \hat{\mathbf{T}} \to G/\Gamma$ to be the map $n \mapsto (\mu n\theta, \nu n\theta, \mu\nu n^2\theta^2/2)\,\Gamma$. Although this does not fit the outline we have given above, our formulas do give *-algebras based on $C_{b,1}(G) \cap C(G/\Gamma)$ which are deformations of $C(G/\Gamma)$. We leave the details to a sequel, but close by showing that it is necessary to use $C_{b,1}(G)$ rather than $B(G)$: $B(G) \cap C(G/\Gamma)$ is not big enough.

EXAMPLE 5.4. Let G be the Heisenberg group based on $\mathbf{R} \times \mathbf{R} \times \mathbf{T}$, as in Example 3.6(2), and let Γ be the subgroup $\{(m, n, 1): m, n \in \mathbf{Z}\}$. Then every function in $B(G) \cap C(G/\Gamma)$ is constant on cosets of the subgroup $K = \{(0, 0, z): \in \mathbf{T}\}$ of G.

Proof. Suppose $\phi \in B(G) \cap C(G/\Gamma)$. There is a representation U of G on Hilbert space $\mathscr{H}$ such that $\phi(x) = (U_x \xi \mid \eta)$. Let P be the orthogonal projection onto $\mathscr{H}_0 = \overline{\text{sp}}\{U_x \eta: x \in G\}$; we may assume that $\xi \in \mathscr{H}_0$ without changing ϕ. For $u \in \Gamma$, we have

$$0 = \phi(xu) - \phi(x) = ((U_{xu} - U_x)\,\xi \mid \eta) = ((U_u - 1)\,\xi \mid U_{x^{-1}}\eta)$$

for all $x \in G$, so $(U_u - 1)\,\xi \perp \mathscr{H}_0$. Since we know $U_u(\xi)$ is in $\mathscr{H}_0$, it follows that $U_u \xi = \xi$ for all $u \in \Gamma$.

By Fourier analysis of the representation $U|_K$, we can expand ξ in a sum $\sum \xi_k$ in which $U_{(0,0,z)}\xi_k = z^k \xi_k$, by taking $\xi_k = \int_K z^{-k} U_{(0,0,z)}\xi\, dz$. Since $(0, 0, z)$ is in the centre of G, we then have $U_u \xi_k = \xi_k$ for all $u \in \Gamma$. Now take $u = (0, 1, 1) \in \Gamma$, $v(x) = (x, 0, 1)$, and $w(x) = (0, 0, \exp 2\pi i x)$. Then we have $uv(x) = v(x)uw(x)$, and hence

$$(U_{v(x)}\xi_k \mid \xi_k) = (U_{uv(x)}\xi_k \mid \xi_k) = (U_{v(x)\,uw(x)}\xi_k \mid \xi_k) = \exp 2\pi i k x (U_{v(x)}\xi_k \mid \xi_k).$$

Thus $(U_{v(x)}\xi_k \mid \xi_k) = 0$ whenever $kx \notin \mathbf{Z}$, so by letting $x \to 0$ we see that $\xi_k = 0$ for $k \neq 0$. Thus $\xi = \xi_0$, $U_{(0,0,z)}\xi = U_{(0,0,z)}\xi_0 = \xi_0 = \xi$, and

$$\phi(x(0, 0, z)) = (U_x U_{(0,0,z)}\xi \mid \eta) = (U_x \xi \mid \eta) = \phi(x),$$

justifying the claim.

APPENDIX: ONE-PARAMETER GROUPS OF DUAL COCYCLES

If $H = \mathbf{R}^m \times \mathbf{T}^n$, then $\hat{H} = \mathbf{R}^m \times \mathbf{Z}^n$, and every cocycle on H is equivalent to one of the form $\sigma(p, q) = \exp i\theta(p, q)$ for some skew-symmetric bilinear form $\theta: \hat{H} \times \hat{H} \to \mathbf{R}$. Each such cocycle lies on a 1-parameter group $\sigma_t: (p, q) \mapsto \exp it\theta(p, q)$, and the Fourier transform of σ_t is then a

1-parameter group of normalised dual cocycles on H. If H is a closed subgroup of a Lie group G, inflating up to G gives a 1-parameter group $\omega(t)$ of normalised dual cocycles on G, which satisfies

$$\Sigma(\omega(t)) = \omega(t)^*. \tag{A1}$$

We shall see in this Appendix that, under some restrictions on the infinitesimal generator, every 1-parameter group of normalised dual cocycles on a connected Lie group G satisfying (A1) is inflated in this way from some such abelian subgroup H.

Suppose $\omega(t)$ is a strongly continuous 1-parameter group of normalised dual cocycles on a connected Lie group G satisfying (A1). By Stone's theorem, there is an (unbounded) self-adjoint operator θ on $L^2(G \times G)$ such that $\omega(t) = \exp it\theta$. Let $\mathscr{U}(\mathfrak{g})$ be the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G, so that $\mathscr{U}(\mathfrak{g})$ consists of the right invariant differential operators on G. We shall assume that the generator θ lies in $\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})$. (In the first paragraph, θ is multiplication by a second degree polynomial on $L^2(\hat{G} \times \hat{G})$, and its Fourier transform is a second order differential operator in $\mathfrak{g} \otimes \mathfrak{g}$.) We shall need three basic facts about $\mathscr{U}(\mathfrak{g})$: that $\mathscr{U}(\mathfrak{g})$ is the linear span of $\{X_1 X_2 \cdots X_k : X_i \in \mathfrak{g}\}$, that δ_G extends to a map $\delta_G : \mathscr{U}(\mathfrak{g}) \to \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})$ satisfying

$$\delta_G(X) = X \otimes 1 + 1 \otimes X \tag{A2}$$

for $X \in \mathfrak{g}$, and that every $X \in \mathscr{U}(\mathfrak{g})$ satisfying (A2) lies in $\mathfrak{g}$. Thus our infinitesimal generator θ maps $L^2(G \times G) \cap C^\infty(G \times G)$ into itself, and the operators $\delta_G \otimes i(\theta)$, $\theta_{12} = \theta \otimes 1$, etc., all lie in $\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})$.

We now look at how the algebraic properties of $\omega(t)$ are reflected in θ. First, differentiating the cocycle identity gives

$$\theta_{12} + \delta_G \otimes i(\theta) = \theta_{23} + i \otimes \delta_G(\theta). \tag{A3}$$

We can similarly differentiate (A1) to obtain

$$\Sigma(\theta) = -\theta \tag{A4}$$

Since $\omega(t)\, \Sigma(\omega(t))^* = \omega(2t)$, the argument of [12, 4.4], which does not use the compactness of G assumed there, gives

$$\omega(2t)_{12}\, \omega(2t)_{13} = \omega(t)_{23}\, i \otimes \delta_G(\omega(2t))\, \omega(-t)_{23}.$$

Differentiating this gives

$$\theta_{12} + \theta_{13} = i \otimes \delta_G(\theta). \tag{A5}$$

The next lemma shows that (A4) and (A5) force θ to lie in the Lie algebra rather than the enveloping algebra. This is at least partly attributable to our assumption (A1): in view of Lemma 1.11, this implies $\nu \otimes \nu(\omega(t)) = \omega(t)$, and hence, formally at least, $\nu \otimes \nu(\theta) = \theta$. Since $\nu(X) = -X$ for $X \in \mathfrak{g}$, this is true if $\theta \in \mathfrak{g} \otimes \mathfrak{g}$, but for more general $T = X_1 X_2 \cdots X_k$ in $\mathscr{U}(\mathfrak{g})$ the relationship between T and $\nu(T) = \nu(X_k) \cdots \nu(X_1)$ could be complicated.

LEMMA A1. *If* $\theta \in \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(g)$ *satisfies Eqs.* (A2) *and* (A5), *then* $\theta \in \mathfrak{g} \otimes \mathfrak{g}$.

Proof. We can write $\theta = \sum_i A_i \otimes B_i$ for some linearly independent set $\{A_i\}$ in $\mathscr{U}(\mathfrak{g})$. Then (A5) gives

$$\sum A_i \otimes B_i \otimes 1 + A_i \otimes 1 \otimes B_i = \sum A_i \otimes \delta_G(B_i),$$

so $\delta_G(B_i) = B_i \otimes 1 + 1 \otimes B_i$, and $B_i \in \mathfrak{g}$. Now use $\theta = -\Sigma(\theta)$ and the same argument to see that $A_i \in \mathfrak{g}$.

If we take the adjoint of the coycle identity for $\omega(t)$, and replace t by $-t$, we obtain

$$\delta_G \otimes i(\omega(t))\, \omega(t)_{12} = i \otimes \delta_G(\omega(t))\, \omega(t)_{23};$$

combining this with the original cocycle identity gives

$$[\omega(t)_{12}, \delta_G \otimes i(\omega(t))] = [\omega(t)_{23}, i \otimes \delta_G(\omega(t))],$$

and differentiating twice gives

$$[\theta_{12}, \delta_G \otimes i(\theta)] = [\theta_{23}, i \otimes \delta_G(\theta)]. \tag{A6}$$

Now putting (A3) and (A5) into (A6) gives

$$[\theta_{12}, \theta_{13} + \theta_{23}] = [\theta_{23}, \theta_{12} + \theta_{13}]. \tag{A7}$$

Next we apply Σ_{12} and Σ_{23} to (A7) to obtain

$$[-\theta_{12}, \theta_{23} + \theta_{13}] = [\theta_{13}, -\theta_{12} + \theta_{23}]$$
$$[\theta_{13}, \theta_{12} - \theta_{23}] = [-\theta_{23}, \theta_{13} + \theta_{12}],$$

and add the last three equations:

$$[\theta_{13}, \theta_{12} - \theta_{23}] = 0. \tag{A8}$$

Finally, putting (A8) into (A7) shows that

$$[\theta_{12}, \theta_{23}] = 0. \tag{A9}$$

Similar arguments show that all the operators θ_{12}, θ_{13}, $i \otimes \delta_G(\theta)$, etc., commute, and hence so do the ones appearing in the cocycle identity.

THEOREM A2. *Suppose G is a connected Lie group, and $\omega(t) = \exp it\theta$ is a one-parameter group of normalised dual cocycles on G whose infinitesimal generator θ is in $\mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})$ and satisfies $\theta = -\Sigma(\theta)$. Then there is a closed connected abelian subgroup H of G such that $\omega(t)$ is inflated from a one-parameter group of normalised dual cocycles on H, as described at the beginning of this Apendix.*

Proof. By Lemma A1, there are $Y_i \in \mathfrak{g}$ and a linearly independent subset $\{X_i\}$ of $\mathfrak{g}$ such that $\theta = \sum X_i \otimes Y_i = -\sum Y_i \otimes X_i$. Equations (A4) and (A9) imply that

$$\sum_{i,j} X_i \otimes [Y_i, Y_j] \otimes X_j = 0.$$

Thus we have $[Y_i, Y_j] = 0$ for all i, j, and $\mathfrak{h} = \mathrm{sp}\{Y_i\}$ is an abelian subalgebra of $\mathfrak{g}$. We may therefore suppose that $\{X_i\}$ is a basis for $\mathfrak{h}$, and that $\theta \in \sum \theta_{ij} X_i \otimes X_j$ with $\theta_{ij} = -\theta_{ji}$. Let H be the connected subgroup of G corresponding to $\mathfrak{h}$; then $\omega(t)$ is inflated from a dual cocycle on H, because its infinitesimal generator θ lies in the Lie algebra $\mathfrak{h} \otimes \mathfrak{h}$. We can view $\hat{H}$ as a subgroup of $\mathfrak{h}$, with the pairing between $\hat{H}$ and H given in terms of the Euclidean product on $\mathfrak{h}$ corresponding to the basis $\{X_i\}$ by $\langle \exp tX, p \rangle = \exp it(X \cdot p)$. Thus $X \in \mathfrak{h}$ is the Fourier transform of the function $p \mapsto iX \cdot p$, $X \otimes Y$ the Fourier transform of $(p, q) \mapsto -(X \cdot p)(Y \cdot q)$, and θ the Fourier transform of

$$(p, q) \mapsto -\sum \theta_{ij}(X_i \cdot p)(X_j \cdot q).$$

We can choose the basis for $\mathfrak{h}$ so that $\mathfrak{h} \cong \mathbf{R}^m \times \mathbf{R}^n$, $H \cong \mathbf{R}^m \times \mathbf{T}^n$, and $\hat{H} \mapsto \mathfrak{h}$ is the usual embedding of $\mathbf{R}^m \times \mathbf{Z}^n$ in $\mathbf{R}^{m+n}$, and then the dual cocycle on H has precisely the form we started with.

Remarks. (1) The assumption that $\theta \in \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})$ is strong, but not unreasonable. We do not know whether we need to assume $\Sigma(\theta) = -\theta$: it is possible that any 1-parameter group $\{\omega(t)\}$ of dual cocycles will be equivalent to one whose generator satsfies this.

(2) Is Theorem A2 also true for a 1-parameter family $\{\omega(t)\}$ of dual cocycles which is not a group? If $\omega(t)$ is normalised and satisfies $\omega(t)^* = \omega(-t)$, $\omega(0) = 1$, then the arguments above show that $\theta = -i\omega'(0)$ still satisfies (A4) and (A5). So if we again assume $\theta \in \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g})$, we deduce that $\theta \in \mathfrak{h} \otimes \mathfrak{h}$ for some abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. (This does not prove the desired theorem, because we cannot in general recover $\omega(t)$ as $\exp it\theta$.) Our observation should be compared with the last part of [25],

where Wassermann argues that a 1-parameter family $\{B(t)\}$ of operators satisfying the classical Yang–Baxter equations always has derivative $b = B'(0)$ coming from an abelian Lie algebra.

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